Dynamical response functions in the quantum Ising chain with a boundary

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We determine dynamical response functions $\langle \mathcal{O}^\dagger(t,x_1)\mathcal{O}(0,x_2)\rangle$ in the scaling limit of the quantum Ising chain on the half line in the presence of a boundary magnetic field. Using a spectral representation in terms of infinite volume form factors and a boundary state, we derive an expansion for the correlator that is found to be rapidly convergent as long as $|\frac{x_1+x_2}{\xi}|\gtrsim 0.2$ where ξ is the correlation length. At sufficiently late times we observe oscillatory behaviour of the correlations arbitrarily far away from the boundary. We investigate the effects of the boundary bound state that is present for a range of boundary magnetic fields.

I. INTRODUCTION

Over the last decade there has been significant progress in the calculation of dynamical correlation functions in integrable quantum field theories[1,2,3,4,5,6]. This has made it possible to determine the dynamic response of a variety of experimentally relevant one-dimensional models of Ising magnets[7,8], integer[9,10] and spin-1/2[11] quantum magnets, Mott insulators[12], two-leg ladders[13], carbon nanotubes[14] and ultra-cold atomic gases[15]. Having established the bulk behaviour of such systems, it is of interest to investigate the effects of impurities on the dynamic response. The simplest possible effect of non-magnetic impurities in a spin chain material is to break the chains into finite segments. Resulting changes in the susceptibility have been investigated in both spin-1/2[16] and spin-1[17] Heisenberg chains as well as two-leg Heisenberg ladders[18]. In the latter two cases the chain break can lead to the formation of edge states, which have been observed in inelastic neutron scattering experiments[19]. Impurity and boundary effects are also relevant to scanning tunneling microscopy experiments[20], where the associated breaking of translational symmetry can lead to fingerprints in the dynamic response that assist in characterising of the underlying bulk behaviour[21]. For the aforementioned reasons it is important to extend the calculation of dynamical response functions in one-dimensional models to systems with boundaries. In the gapless case this is readily accomplished by boundary conformal field theory[22,23], while the gapful case is as usual more involved. One successful approach has been the extension of the truncated conformal space approach[24] to systems with boundary[25,26]. The method we will follow in the present work is the extension of the form factor bootstrap approach to boundary integrable field theories[27,28,29,30,31]. We will focus on the simplest integrable field theory, the quantum Ising chain[6,32,33,34,35,36,37,38]. One-point functions in the boundary Ising model have been studied by Konik *et al.*[29]. The purpose of the present work is to analyse two-point functions relevant to scattering experiments.

This article is organised as follows. First, we will review the basic facts on the Ising field theory in the bulk. We proceed by discussing the Ising model on the half line in the presence of a boundary magnetic field and the form factor bootstrap approach to boundary integrable field theories. In order to set the stage, we calculate the Green's functions of the Majorana fermions, which show typical light-cone effects. After a brief discussion of the local magnetisation we proceed with our main result, the calculation of the dynamical spin-spin correlations and the corresponding spectral function. The derived expansion in terms of infinite volume form factors and a boundary state is found to be rapidly convergent as long as $|\frac{x_1+x_2}{\xi}|\gtrsim 0.2$ where ξ is the correlation length. As for the Green's functions we observe light-cone effects, which result in oscillatory behaviour of the correlations arbitrarily far away from the boundary. Furthermore, we study the effects of the boundary bound state, which exists for sufficiently small values of the boundary magnetic field, on the correlation function. Finally, we discuss the two-point function of the disorder field, which shows the same qualitative features as the spin-spin correlations.

II. QUANTUM ISING CHAIN

The Hamiltonian of one-dimensional quantum Ising model is given by

$$H_{\mathrm{latt}} = -J\sum_i \big(\sigma_i^{\mathrm{z}}\,\sigma_{i+1}^{\mathrm{z}} + \lambda\,\sigma_i^{\mathrm{x}}\big). \tag{1}$$

Here σ^{x} and σ^{z} are the Pauli matrices and J is the exchange energy. The model (1) is related to the two-dimensional classical Ising model, see for example Ref. 39. The critical temperature of the latter is related to the coupling constant of the quantum Ising chain by $\lambda=\frac{T-T_{\mathrm{c}}}{T_{\mathrm{c}}}$. The Hamiltonian (1) is invariant under the $\mathbb{Z}_2$-transformation $\sigma_i^{\mathrm{x}}\to\sigma_i^{\mathrm{x}}$,

$\sigma_i^z \to -\sigma_i^z$. For $\lambda < 1$, which corresponds to the low-temperature phase of the 2D classical Ising model, this symmetry is broken. The order parameter field σ_i^z takes a non-zero expectation value and the ground state is two-fold degenerate. For $\lambda > 1$ the model has a unique ground state. This regime corresponds to the disordered high-temperature phase of the classical 2D Ising model. The point $\lambda = 1$ is the location of a quantum phase transition.

At small deviations from criticality, $|\lambda - 1| \ll 1$, one can pass to the continuum limit[39,40]. This leads to the Ising field theory, which is defined by the Euclidean action[6,40]

$$\mathcal{S} = \frac{1}{2\pi} \int d^2x \left(\psi \bar{\partial} \psi + \bar{\psi} \partial \bar{\psi} + \mathrm{i}\, M \bar{\psi} \psi \right), \tag{2}$$

where ψ and $\bar{\psi}$ are the two components of a Majorana fermion. The complex coordinates used in (2) are given by $z = \tau + \mathrm{i}\, x$ and $\bar{z} = \tau - \mathrm{i}\, x$, where $\tau = \mathrm{i}\, t$ denotes imaginary time. This results in $\partial = \partial_z = (\partial_\tau - \mathrm{i}\, \partial_x)/2$ and $\bar{\partial} = \partial_{\bar{z}} = (\partial_\tau + \mathrm{i}\, \partial_x)/2$. The mass is proportional to the distance from the critical point, $M \propto J(1 - \lambda)$. The model (2) is conformally invariant at the critical point $M = 0$ (see for example Ref. 41). In the ordered phase, which we will consider throughout this paper, the mass is positive. Furthermore, we set the velocity to one and use the short distance normalisations

$$z \, \langle 0 | \, \psi(\tau, x) \, \psi(0,0) \, |0\rangle \to 1, \quad |z|^{1/4} \, \langle 0 | \, \sigma^z(\tau, x) \, \sigma^z(0,0) \, |0\rangle \to 1, \quad \text{as} \quad |z| \to 0. \tag{3}$$

For later use we will need the notion of mutual semi-locality of operators[1,3,6,38]. Let us consider the operator product $O_1(\tau, x) \, O_2(0,0) = O_1(z, \bar{z}) \, O_2(0,0)$. If we take O_1 counterclockwise around O_2 in the plane, *i.e.,* we perform the analytic continuation $z \to e^{2\pi \mathrm{i}} \, z$, $\bar{z} \to e^{-2\pi \mathrm{i}} \, \bar{z}$, the operators O_1 and O_2 are said to be mutually semi-local if

$$O_1(e^{2\pi \mathrm{i}} \, z, e^{-2\pi \mathrm{i}} \, \bar{z}) \, O_2(0,0) = l_{O_1 O_2} \, O_1(z, \bar{z}) \, O_2(0,0). \tag{4}$$

The phase $l_{O_1 O_2}$ is called the semi-locality factor. The two fields are mutually local if $l_{O_1 O_2} = 1$. Semi-locality is the mildest form of non-locality, in general the right-hand side of (4) may be more complicated. The mutual semi-locality factor of the spin and disorder operators can be read off from their operator product expansion[6,41]

$$\sigma^z(z, \bar{z}) \, \mu^z(0,0) \sim \frac{1}{\sqrt{2} \, |z|^{1/4}} \left(e^{\mathrm{i}\, \pi/4} \sqrt{z} \, \psi(0) + e^{-\mathrm{i}\, \pi/4} \sqrt{\bar{z}} \, \bar{\psi}(0) \right). \tag{5}$$

This implies that when taking σ^z once around μ^z one obtains an extra minus sign, *i.e.,* $l_{\sigma^z \mu^z} = -1$. In the same way one finds $l_{\psi \mu^z} = l_{\bar{\psi} \mu^z} = l_{\psi \sigma^z} = l_{\bar{\psi} \sigma^z} = -1$. On the other hand, the disorder field μ^z is local with respect to itself.

In order proceed further, we need to construct a basis of scattering states. In order to do so, we need to specify which "fundamental field" (that is a field that has a non-vanishing matrix element between the vacuum and the one-particle states) we will use to create the fundamental excitations. In order to avoid additional signs in many of the equations it is customary to choose the fundamental field to be bosonic. We will therefore use the disorder field μ^z. This implies that the fundamental excitations are viewed as bosons with two-particle scattering matrix $S = -1$. Let us denote the corresponding annihilation and creation operators by $A(\theta)$ and $A^\dagger(\theta)$ respectively. They fulfil the Faddeev–Zamolodchikov algebra

$$\begin{aligned} A(\theta_1) A(\theta_2) &= S A(\theta_2) A(\theta_1), \\ A^\dagger(\theta_1) A^\dagger(\theta_2) &= S A^\dagger(\theta_2) A^\dagger(\theta_1), \\ A(\theta_1) A^\dagger(\theta_2) &= 2\pi \delta(\theta_1 - \theta_2) + S A^\dagger(\theta_2) A(\theta_1), \end{aligned} \tag{6}$$

where the scattering matrix is $S = -1$. The vacuum state is then defined by

$$A(\theta)|0\rangle = 0, \tag{7}$$

and a basis of scattering states is given as

$$|\theta_1, \ldots, \theta_n\rangle = A^\dagger(\theta_1) \ldots A^\dagger(\theta_n) \, |0\rangle \, . \tag{8}$$

In terms of the states (8) the resolution of the identity reads

$$\mathrm{id} = |0\rangle \, \langle 0| + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{-\infty}^{\infty} \frac{d\theta_1 \ldots d\theta_n}{(2\pi)^n} \, |\theta_1, \ldots, \theta_n\rangle \, \langle \theta_n, \ldots, \theta_1| \, . \tag{9}$$

For the calculation of correlation functions the knowledge of the matrix elements or form factors of local operators is necessary. In the ordered phase the non-vanishing form factors of σ^{z} contain an even number of particles and are explicitly given by[6,35,37,38]

$$f(\theta_1,\ldots,\theta_{2n}) = \langle 0|\,\sigma^{\mathrm{z}}(0,0)\,|\theta_1,\ldots,\theta_{2n}\rangle = \mathrm{i}^{\,n}\sigma_0 \prod_{\substack{i,j=1\\ i<j}}^{2n} \tanh\frac{\theta_i-\theta_j}{2}, \tag{10}$$

where $\sigma_0 = \langle 0|\,\sigma^{\mathrm{z}}\,|0\rangle$. For comparison, the form factors of the disorder operator μ^{z} are only non-vanishing if the number of particles is odd,

$$\langle 0|\,\mu^{\mathrm{z}}(0,0)\,|\theta_1,\ldots,\theta_{2n+1}\rangle = \mathrm{i}^{\,n}\sigma_0 \prod_{\substack{i,j=1\\ i<j}}^{2n+1} \tanh\frac{\theta_i-\theta_j}{2}. \tag{11}$$

The form factors (10) and (11) follow from the following set of requirements[1,3,6]:

1. The form factors $f(\theta_1,\ldots,\theta_n)$ are meromorphic functions of θ_n in the physical strip $0 \le \mathfrak{Im}\,\theta_n \le 2\pi$. There exist only simple poles in this strip.
2. Scattering axiom:
$$f(\theta_1,\ldots,\theta_{i+1},\theta_i,\ldots,\theta_n) = S\,f(\theta_1,\ldots,\theta_i,\theta_{i+1},\ldots,\theta_n),$$
with the scattering matrix $S=-1$.
3. Periodicity axiom:
$$f(\theta_1+2\pi\mathrm{i}\,,\theta_2,\ldots,\theta_n) = l_{O\mu^{\mathrm{z}}}\,f(\theta_2,\ldots,\theta_n,\theta_1),$$
where $O=\sigma^{\mathrm{z}}$ or μ^{z}. The mutual non-locality phases are given by $l_{\sigma^{\mathrm{z}}\mu^{\mathrm{z}}}=-1$ and $l_{\mu^{\mathrm{z}}\mu^{\mathrm{z}}}=1$, respectively.
4. Lorentz invariance:
$$f(\theta_1+\alpha,\ldots,\theta_n+\alpha) = e^{s\alpha}\,f(\theta_1,\ldots,\theta_n),$$
where s denotes the spin of the fields, which is $s_{\sigma^{\mathrm{z}}}=0$ and $s_{\mu^{\mathrm{z}}}=0$ as well as $s_\psi=-1/2$ and $s_{\bar\psi}=1/2$.
5. Annihilation pole axiom:
$$\mathrm{Res}\big[f(\theta',\theta,\theta_1,\ldots,\theta_n),\theta'=\theta+\mathrm{i}\,\pi\big] = \mathrm{i}\,f(\theta_1,\ldots,\theta_n)\left[1-l_{O\mu^{\mathrm{z}}}\textstyle\prod_{i=1}^{n}S\right].$$
Note that the squared brackets on right-hand side equal 2 for σ^{z} as well as μ^{z}, as the extra minus sign due to $l_{\sigma^{\mathrm{z}}\mu^{\mathrm{z}}}=-1$ is compensated by an additional factor S. As there exist no bound states in the Ising model, these are the only poles of the form factors.

We note that if we had used the Majorana fermion ψ as fundamental field, the excitations would be viewed as fermions with unity scattering matrix. Furthermore, there would appear additional minus signs in the axioms above[38].

III. BOUNDARY ISING MODEL

We now turn to the Ising field theory on the half-plane (τ,x), $\tau\in\mathbb{R}$, $x\in(-\infty,0]$. The boundary is located at $x=0$ and τ denotes imaginary time ($\tau=\mathrm{i}\,t$). The Hilbert space of states associated with the semi-infinite line $\tau=\text{const.}$, $-\infty<x\le 0$, is denoted by $\mathcal{H}_{\mathrm{B}}$. The boundary condition is imposed through application of an external time-independent magnetic field h which couples to the boundary spin $\sigma^{\mathrm{z}}_{\mathrm{b}}(\tau)=\sigma^{\mathrm{z}}(\tau,0)$. The action for this system is given by[27]

$$\mathcal{S} = \frac{1}{2\pi}\int d\tau\int_{-\infty}^{0}dx\,\big(\psi\bar\partial\psi+\bar\psi\partial\bar\psi+\mathrm{i}\,M\bar\psi\psi\big)+\frac{1}{2\pi}\int d\tau\left(h\sigma^{\mathrm{z}}_{\mathrm{b}}-\frac{\mathrm{i}}{2}\bar\psi\psi-\frac{1}{2}\,a\,\partial_\tau a\right). \tag{12}$$

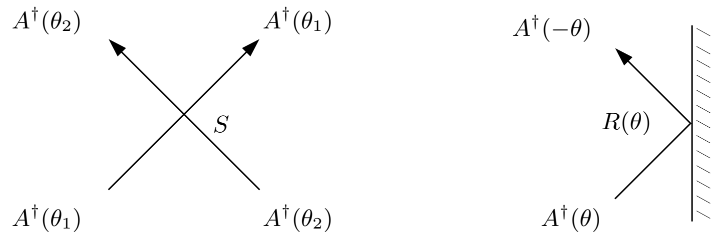

FIG. 1: Two-particle scattering and scattering off the boundary.

It was shown by Affleck and Ludwig[42] that in the critical model the application a magnetic field at the boundary generates a flow from free to fixed boundary conditions. The properties of the boundary spin operator $\sigma^{\mathrm{z}}_{\mathrm{b}}$ were analysed in Refs 23,43. In the Lagrangian framework defined by (12) it can be written as[27]

$$\sigma^{\mathrm{z}}_{\mathrm{b}}(\tau) = \frac{1}{2}\big(\psi(\tau,x)+\bar{\psi}(\tau,x)\big)\Big|_{x=0}\, a(\tau). \tag{13}$$

Here $a(\tau)$ is an additional fermionic boundary degree of freedom which is introduced to describe the ground-state degeneracy and anticommutes with ψ and $\bar{\psi}$. The cases $h=0$ and $h\to\infty$ correspond to free and fixed boundary conditions, respectively. We note that the application of a non-zero boundary magnetic field removes the ground-state degeneracy of the Ising model.

In terms of the Majorana fermions the boundary condition reads

$$-\,\mathrm{i}\,\frac{d}{d\tau}\big(\psi-\bar{\psi}\big)\Big|_{x=0} = \frac{h^2}{2}\big(\psi+\bar{\psi}\big)\Big|_{x=0}. \tag{14}$$

As was shown by Ghoshal and Zamolodchikov[27] the Ising model with boundary conditions (14) still possesses infinitely many integrals of motion (which can be constructed from the integrals of motion of the model in the bulk) and hence remains integrable.

If one thinks of the boundary as an infinitely heavy impenetrable particle B sitting at $x=0$ and writes $|0_{\mathrm{B}}\rangle = B\,|0\rangle \in \mathcal{H}_{\mathrm{B}}$ for the ground state in the presence of the boundary, the scattering of particles created by the operators $A^\dagger(\theta)$ at the boundary is encoded in the relations

$$A^\dagger(\theta)B = R(\theta)A^\dagger(-\theta)B. \tag{15}$$

The function $R(\theta)$ is interpreted as the single-particle reflection amplitude off the boundary. In order to preserve the integrability of the corresponding bulk system, the boundary scattering matrix $R(\theta)$ has to satisfy several conditions as discussed in Ref. 27, which leads to the following result for the boundary scattering matrix for the Ising model with boundary magnetic field h

$$R(\theta) = \mathrm{i}\,\tanh\!\left(\frac{\mathrm{i}\,\pi}{4}-\frac{\theta}{2}\right)\frac{\kappa-\mathrm{i}\,\sinh\theta}{\kappa+\mathrm{i}\,\sinh\theta},\quad \kappa = 1-\frac{h^2}{2M}. \tag{16}$$

Free boundary conditions are recovered for $h=0$, whereas fixed boundary conditions are obtained in the limit $h\to\infty$ (we restrict ourselves to $h\geq 0$). The purpose of the present work is to calculate the two-point correlation function

$$C(\tau,x_1,x_2) = \langle 0_{\mathrm{B}}|\,\mathcal{T}_\tau\,\sigma^{\mathrm{z}}(\tau,x_1)\,\sigma^{\mathrm{z}}(0,x_2)\,|0_{\mathrm{B}}\rangle\,, \tag{17}$$

where $\mathcal{T}_\tau$ is the time-ordering operator. The time-dependence of the operators σ^{z} is given by

$$\sigma^{\mathrm{z}}(\tau,x) = e^{\tau H_{\mathrm{B}}}\,\sigma^{\mathrm{z}}(0,x)\,e^{-\tau H_{\mathrm{B}}}, \tag{18}$$

where H_{B} is the Hamiltonian of the system in the presence of the boundary.

As we are working in the Euclidean formalism, τ and x are interchangeable. Therefore, one may also take x to be the Euclidean time, which implies that the equal-time section is the infinite line, $x=$ const., $-\infty<\tau<\infty$, and the associated Hilbert space $\mathcal{H}$ is that of the corresponding bulk theory (8). The boundary at $x=0$ now appears as an initial condition which is expressed in terms of a "boundary state" $|\mathrm{B}\rangle$. As was shown by Ghoshal and Zamolodchikov[27], the correlation function (17) can then be expressed as

$$C(\tau,x_1,x_2) = \frac{\langle 0|\,\mathcal{T}_x\,\sigma^{\mathrm{z}}(\tau,x_1)\,\sigma^{\mathrm{z}}(0,x_2)\,|\mathrm{B}\rangle}{\langle 0|\,\mathrm{B}\rangle}. \tag{19}$$

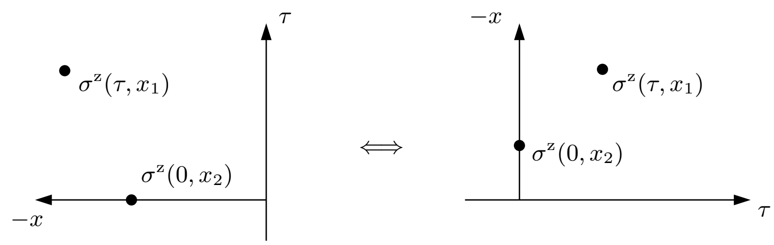

FIG. 2: Euclidean rotation. The boundary condition at $x = 0$ turns into an initial condition. The "time" in the rotated system runs between $x = 0$ and $x = -\infty$.

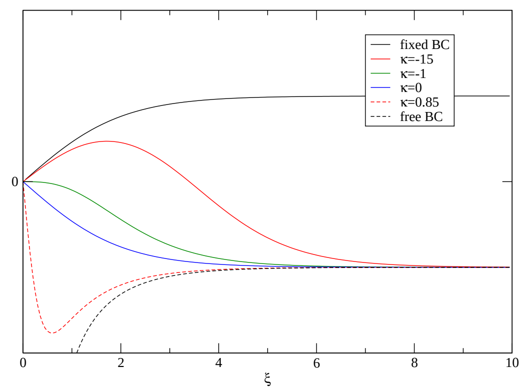

FIG. 3: $\hat{K}(\xi)$ as function of ξ. Note that $h = 2\sqrt{M}$ corresponds to $\kappa = -1$ and $h = h_c = \sqrt{2M}$ to $\kappa = 0$.

Here $\mathcal{T}_x$ is the x-ordering operator, which orders the largest x_i to the right, and $|0\rangle \in \mathcal{H}$ is the ground state of the model on the infinite line. The boundary state is given by

$$|\mathrm{B}\rangle = \exp\left(\int_0^\infty \frac{d\xi}{2\pi} K(\xi) A^\dagger(-\xi) A^\dagger(\xi)\right)|0\rangle\,, \tag{20}$$

where $K(\xi) = R(\mathrm{i}\,\pi/2 - \xi)$. For the Ising model it is explicitly given by

$$K(\xi) = \mathrm{i}\,\tanh\frac{\xi}{2}\,\frac{\kappa + \cosh\xi}{\kappa - \cosh\xi}. \tag{21}$$

This simplifies to

$$K_{\text{free}}(\xi) = -\mathrm{i}\,\coth\frac{\xi}{2}, \quad K_{\text{fixed}}(\xi) = \mathrm{i}\,\tanh\frac{\xi}{2}, \tag{22}$$

for free and fixed boundary conditions, respectively. For later use we introduce the real function $\hat{K}(\xi) = -\mathrm{i}\,K(\xi)$, which is plotted for several values of κ in Fig. 3. We note that for $0 \le h \le 2\sqrt{M}$ the function $\hat{K}(\xi)$ is negative for all ξ, whereas for $2\sqrt{M} < h$ the function $\hat{K}(\xi)$ is positive in a certain ξ-interval. Furthermore, for $0 < h < h_c = \sqrt{2M}$ one finds $\hat{K}(\xi) < -1$ for certain ξ. In this region the boundary scattering matrix (16) has a pole in the strip $0 < \mathfrak{Im}\,\xi < \pi/2$, indicating the existence of a boundary bound state[27]. We will discuss this state in more detail in Sec. III A below. Finally, we note that for free boundary conditions the boundary scattering matrix (16) has a pole at $\xi = 0$. This results in the appearance of a zero-momentum mode in the boundary state, *i.e.,* (20) is replaced by[27]

$$|\mathrm{B}_{\text{free}}\rangle = \big(1 + A^\dagger(0)\big)\exp\left(\int_0^\infty \frac{d\xi}{2\pi} K(\xi) A^\dagger(-\xi) A^\dagger(\xi)\right)|0\rangle\,. \tag{23}$$

Since all form factors of σ^{z} involving an odd number of particles vanish, this zero-momentum mode does not contribute to the two-point function (19) of σ^{z}, and hence will be ignored in our analysis. Finally, we note that $K(\xi) = -K(-\xi)$ and $K(\xi) \to 0$ $(\xi \to 0)$ except for free boundary conditions.

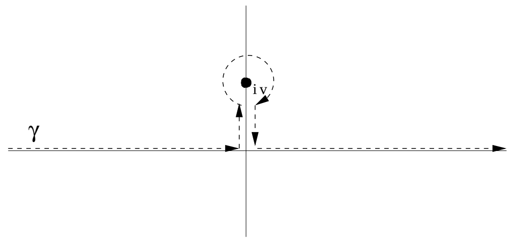

FIG. 4: Contour of integration for the excited boundary state.

As we have interchanged space and time and x is running from 0 to $-\infty$ in the new framework, the τ- and x-dependence of operators $\sigma^{\mathrm{z}}(\tau,x)$ is now given by

$$\sigma^{\mathrm{z}}(\tau,x)=e^{-xH}\,e^{-\mathrm{i}\,\tau P}\,\sigma^{\mathrm{z}}(0,0)\,e^{\mathrm{i}\,\tau P}\,e^{xH}, \tag{24}$$

where H is the Hamiltonian of the system on the infinite line, $-\infty<\tau<\infty$, and P is the total momentum.

A. Boundary bound state

If we consider free boundary conditions ($h=0$) there exist two degenerate ground states $|0_{\mathrm{B}},\pm\rangle$. The effect of a small field $0<h<h_{\mathrm{c}}=\sqrt{2M}$ is to split these into two non-degenerate states $|0_{\mathrm{B}}\rangle$ and $|1_{\mathrm{B}}\rangle$, where $|1_{\mathrm{B}}\rangle$ can be interpreted as a boundary bound state[27]. In this domain we can parametrise κ as

$$\kappa=\cos v,\quad 0<v<\frac{\pi}{2}. \tag{25}$$

The two states can be distinguished by the asymptotic behaviour of the one-point function $\langle 0_{\mathrm{B}}|\,\sigma^{\mathrm{z}}(x)\,|0_{\mathrm{B}}\rangle\to+\sigma_0$ or $\langle 1_{\mathrm{B}}|\,\sigma^{\mathrm{z}}(x)\,|1_{\mathrm{B}}\rangle\to-\sigma_0$ for $x\to-\infty$, respectively (we assume that $h\geq 0$). The energy of $|1_{\mathrm{B}}\rangle$ is given by $E_1=E_0+M\sin v$, where E_0 denotes the ground-state energy. If h approaches the critical value h_{c}, the boundary bound state becomes weakly bound and its effective size diverges. For $h>h_{\mathrm{c}}$ (16) possesses no pole in the physical strip and hence no boundary bound state occurs.

Correlation functions in the boundary bound state can be calculated following Ref. 27

$$\langle 1_{\mathrm{B}}|\,\mathcal{T}_\tau\,\sigma^{\mathrm{z}}(\tau,x_1)\,\sigma^{\mathrm{z}}(0,x_2)\,|1_{\mathrm{B}}\rangle=\frac{\langle 0'|\,\mathcal{T}_x\,\sigma^{\mathrm{z}}(\tau,x_1)\,\sigma^{\mathrm{z}}(0,x_2)\,|\mathrm{B}'\rangle}{\langle 0'|\,\mathrm{B}'\rangle}, \tag{26}$$

where $|0'\rangle=|0,-\rangle\in\mathcal{H}$ denotes the "wrong" ground state. The excited boundary state is given by[27]

$$|\mathrm{B}'\rangle=\exp\left(\frac{1}{2}\int_\gamma\frac{d\xi}{2\pi}K(\xi)A^\dagger(-\xi)A^\dagger(\xi)\right)|0'\rangle\,, \tag{27}$$

where the contour of integration is shown in Fig. 4. The contour encircles the pole of (21) at $\xi=\mathrm{i}\,v$, whose residue equals $-2\mathrm{i}\cot v\tan(v/2)$.

Due to the $\mathbb{Z}_2$-invariance of the Ising model in the bulk the matrix elements of σ^{z} remain unchanged up to a minus sign when considering the state $|0'\rangle$, *i.e.,* the form factors are

$$\langle 0'|\,\sigma^{\mathrm{z}}\,|\{\theta_1,\ldots,\theta_{2n}\}'\rangle\equiv\langle 0'|\,\sigma^{\mathrm{z}}\,A^\dagger(\theta_1)\ldots A^\dagger(\theta_{2n})\,|0'\rangle=-\mathrm{i}^{\,n}\sigma_0\prod_{\substack{i,j=1\\ i<j}}^{2n}\tanh\frac{\theta_i-\theta_j}{2}. \tag{28}$$

In particular, we have $E_0^{\mathrm{bulk}}=E_{0'}^{\mathrm{bulk}}$. The minus sign can be deduced from $\langle 0'|\,\sigma^{\mathrm{z}}\,|0'\rangle=-\langle 0|\,\sigma^{\mathrm{z}}\,|0\rangle$.

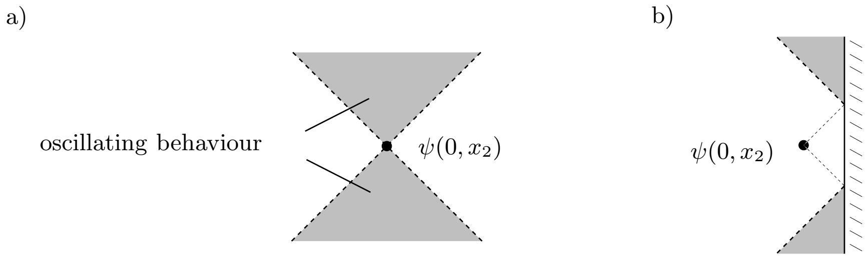

FIG. 5: Light-cone effects observed for the Green's function (31). a) In the bulk one observes oscillating behaviour if $r^2 < t^2$. b) The boundary contributions show oscillating behaviour if $4R^2 < t^2$.

IV. GREEN'S FUNCTIONS

In order to set the stage, we first calculate the Green's functions of the Majorana fermions. The mode expansions in the presence of the boundary and in the absence of a boundary bound state are given by

$$\psi(\tau,x) = \sqrt{\frac{M}{2}} \int_0^\infty \frac{d\theta}{\sqrt{2\pi}} \Big[c(\theta)\, e^{-\frac{\mathrm{i}\pi}{4}}\, e^{-M\tau\cosh\theta} \Big(e^{-\frac{\theta}{2}}\, e^{\mathrm{i}\,Mx\sinh\theta} + R(\theta)\, e^{\frac{\theta}{2}}\, e^{-\mathrm{i}\,Mx\sinh\theta} \Big) \\ + c^\dagger(\theta)\, e^{\frac{\mathrm{i}\pi}{4}}\, e^{M\tau\cosh\theta} \Big(e^{-\frac{\theta}{2}}\, e^{-\mathrm{i}\,Mx\sinh\theta} + R(-\theta)\, e^{\frac{\theta}{2}}\, e^{\mathrm{i}\,Mx\sinh\theta} \Big) \Big], \tag{29}$$

$$\bar\psi(\tau,x) = \sqrt{\frac{M}{2}} \int_0^\infty \frac{d\theta}{\sqrt{2\pi}} \Big[c(\theta)\, e^{\frac{\mathrm{i}\pi}{4}}\, e^{-M\tau\cosh\theta} \Big(e^{\frac{\theta}{2}}\, e^{\mathrm{i}\,Mx\sinh\theta} + R(\theta)\, e^{-\frac{\theta}{2}}\, e^{-\mathrm{i}\,Mx\sinh\theta} \Big) \\ + c^\dagger(\theta)\, e^{-\frac{\mathrm{i}\pi}{4}}\, e^{M\tau\cosh\theta} \Big(e^{\frac{\theta}{2}}\, e^{-\mathrm{i}\,Mx\sinh\theta} + R(-\theta)\, e^{-\frac{\theta}{2}}\, e^{\mathrm{i}\,Mx\sinh\theta} \Big) \Big]. \tag{30}$$

Here $c(\theta)$ and $c^\dagger(\theta)$ are canonical fermion annihilation and creation operators $\{c(\theta), c^\dagger(\theta')\} = 2\pi\delta(\theta-\theta')$.

The Green's function of ψ is now easily derived to be (we assume $\tau > 0$ for simplicity)

$$\begin{aligned} \langle 0_\mathrm{B}|\, \psi(\tau,x_1)\, \psi(0,x_2)\, |0_\mathrm{B}\rangle &= \frac{M}{2} \int_{-\infty}^{\infty} d\theta\, e^{-M\tau\cosh\theta} \Big[e^{-\theta}\, e^{\mathrm{i}\,Mr\sinh\theta} + R(\theta)\, e^{-2M\mathrm{i}\,R\sinh\theta} \Big] \\ &= M \sqrt{\frac{\mathrm{i}\,\tau - r}{\mathrm{i}\,\tau + r}}\, K_1\big(M\sqrt{r^2+\tau^2}\big) + \frac{M}{2} \int_{-\infty}^{\infty} d\theta\, R\big(\theta + \mathrm{i}\,\theta_0\big)\, e^{-M\sqrt{4R^2+\tau^2}\cosh\theta}, \end{aligned} \tag{31}$$

where $\theta_0 = \arctan\big(2|R|/\tau\big)$ and K_1 denotes the modified Bessel function of order one[44]. Furthermore, we have introduced centre-of-mass coordinates $R = (x_1+x_2)/2$ and $r = x_2 - x_1$. We stress that $R \leq 0$. In real space, $\tau = \mathrm{i}\,t$, the first term is oscillating for $r^2 < t^2$ and damped for $r^2 > t^2$, *i.e.* we observe a light-cone effect. The second term is oscillating for $4R^2 < t^2$ and damped otherwise. The physical interpretation of the oscillating behaviour is that for $4R^2 < t^2$ a particle can propagate from $(0,x_2)$ to (t,x_1) via the boundary (see Fig. 5). If one calculates the Green's function using the rotated system and the boundary state, one obtains an additional phase of $\pi/2$. The physical origin of this phase is the Lorentz spin $s_\psi = -1/2$ of ψ, which implies that the Green's function transforms non-trivial under Lorentz rotations. In the case $h < h_\mathrm{c}$ there is an additional term due to the presence of the boundary bound state.

In the same way one obtains (we assume $\tau > 0$)

$$\langle 0_\mathrm{B}|\, \psi(\tau,x_1)\, \bar\psi(0,x_2)\, |0_\mathrm{B}\rangle = -\mathrm{i}\,M\, K_0\big(M\sqrt{r^2+\tau^2}\big) - \mathrm{i}\,\frac{M}{2}\sqrt{\frac{\mathrm{i}\,\tau + 2R}{\mathrm{i}\,\tau - 2R}} \int_{-\infty}^{\infty} d\theta\, R\big(\theta + \mathrm{i}\,\theta_0\big)\, e^{\theta}\, e^{-M\sqrt{4R^2+\tau^2}\cosh\theta}, \tag{32}$$

which shows the same kind of oscillatory behaviour as (31).

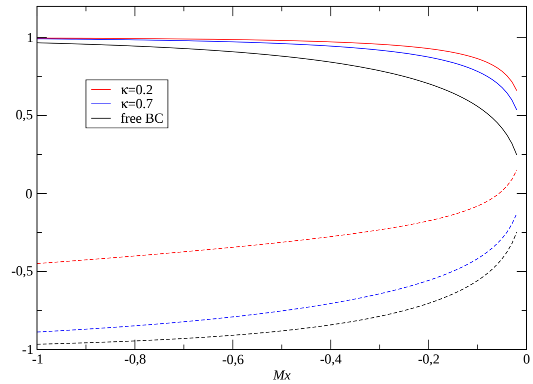

FIG. 6: Local magnetisation in the ground state and the boundary bound state for different values of the boundary magnetic field $h>0$ up to first order in the boundary reflection matrix K. The full lines represent the ground-state result (33), the dashed lines the result for the boundary bound state (34).

V. LOCAL MAGNETISATION

In this section we briefly discuss the one-point function of the spin operator. The one-point function in the ground state $|0_{\rm B}\rangle$ was first calculated by Konik, LeClair and Mussardo[29] and to second order in K is given by

$$\begin{aligned}\langle 0_{\rm B}|\,\sigma^{\rm z}(x)\,|0_{\rm B}\rangle &= \sigma_0 - {\rm i}\,\sigma_0\int_0^\infty \frac{d\xi}{2\pi}\,K(\xi)\,\tanh\xi\,e^{2Mx\cosh\xi}\\ &\quad -\frac{\sigma_0}{2}\int_0^\infty\frac{d\xi_1 d\xi_2}{(2\pi)^2}\prod_{i=1}^2 K(\xi_i)\tanh\xi_i\left(\frac{\cosh\xi_1-\cosh\xi_2}{\cosh\xi_1+\cosh\xi_2}\right)^2 e^{2Mx\sum_i\cosh\xi_i}+\ldots\end{aligned}\tag{33}$$

Here $x<0$ denotes the distance from the boundary. The full series can be written as Fredholm determinant[29].

Using (26), (27) and (28), we can calculate the expectation value of the spin operator in the boundary bound state

$$\begin{aligned}\langle 1_{\rm B}|\,\sigma^{\rm z}(x)\,|1_{\rm B}\rangle &= -\sigma_0 + {\rm i}\,\frac{\sigma_0}{2}\int_\gamma \frac{d\xi}{2\pi}\,K(\xi)\,\tanh\xi\,e^{2Mx\cosh\xi}+\ldots\\ &= -\sigma_0 + {\rm i}\,\sigma_0\int_0^\infty \frac{d\xi}{2\pi}\,K(\xi)\,\tanh\xi\,e^{2Mx\cosh\xi}+\sigma_0\,\tan\frac{v}{2}\,e^{2Mx\cos v}+\ldots,\end{aligned}\tag{34}$$

where the contour γ is defined in Fig. 4 and we recall that $\kappa = 1-h^2/2M=\cos v$. We observe that $\langle 1_{\rm B}|\,\sigma^{\rm z}(x)\,|1_{\rm B}\rangle\to -\sigma_0$ for $x\to-\infty$ as expected. In the limit $v\to\pi/2$ the last term remains comparable to σ_0 even for large distances x to the boundary. In this limit the boundary bound state becomes weakly bound and its size diverges. The local magnetisations of the ground state and the boundary bound state are shown in Fig. 6. For free boundary conditions the two results equal each other up to a global minus sign. For finite boundary magnetic field the spins in the vicinity of the boundary are aligned parallel to h, which increases the local magnetisation.

VI. SPIN-SPIN CORRELATION FUNCTION

We now turn to the calculation the leading terms in an expansion in powers of the boundary reflection matrix K of the two-point function of Ising spins (17). Using a spectral representation of the correlator in terms of the scattering states (8) we obtain the following expression

$$C(\tau,x_1,x_2)=\sum_{n=0}^\infty\sum_{m=0}^\infty C_{2n\,2m}(\tau,x_1,x_2),\tag{35}$$

where

$$C_{2n\,2m}(\tau,x_1,x_2) = \frac{1}{m!}\frac{1}{(2n)!}\int_0^\infty \frac{d\xi_1\dots d\xi_m}{(2\pi)^m}\int_{-\infty}^\infty \frac{d\theta_1\dots d\theta_{2n}}{(2\pi)^{2n}}\,K(\xi_1)\dots K(\xi_m) \cdot \begin{cases} \langle 0|\,\sigma^{\mathrm{z}}(\tau,x_1)\,|\theta_1,\dots,\theta_{2n}\rangle\,\langle\theta_{2n},\dots,\theta_1|\,\sigma^{\mathrm{z}}(0,x_2)\,|{-\xi_1},\xi_1,\dots,-\xi_m,\xi_m\rangle, & x_1<x_2,\\ \langle 0|\,\sigma^{\mathrm{z}}(0,x_2)\,|\theta_1,\dots,\theta_{2n}\rangle\,\langle\theta_{2n},\dots,\theta_1|\,\sigma^{\mathrm{z}}(\tau,x_1)\,|{-\xi_1},\xi_1,\dots,-\xi_m,\xi_m\rangle, & x_1>x_2.\end{cases} \tag{36}$$

We label the various terms in the double expansion (35) by the numbers of particles in the intermediate state $2n$ and in the boundary state $2m$, respectively. The connected correlator is given by

$$C_{\mathrm{conn}}(\tau,x_1,x_2) = \langle 0_{\mathrm{B}}|\,\mathcal{T}_\tau\,\delta\sigma^{\mathrm{z}}(\tau,x_1)\,\delta\sigma^{\mathrm{z}}(0,x_2)\,|0_{\mathrm{B}}\rangle = C(\tau,x_1,x_2) - \langle 0_{\mathrm{B}}|\,\sigma^{\mathrm{z}}(x_1)\,|0_{\mathrm{B}}\rangle\,\langle 0_{\mathrm{B}}|\,\sigma^{\mathrm{z}}(x_2)\,|0_{\mathrm{B}}\rangle\,, \tag{37}$$

where $\delta\sigma^{\mathrm{z}}(\tau,x) = \sigma^{\mathrm{z}}(\tau,x) - \langle 0_{\mathrm{B}}|\,\sigma^{\mathrm{z}}(\tau,x)\,|0_{\mathrm{B}}\rangle$ and the second term is in fact independent of τ.

A. Regularisation

The functions (36) contain matrix elements of the form

$$\langle\theta_1,\dots,\theta_n|\,\sigma^{\mathrm{z}}\,|\xi_1,\dots,\xi_m\rangle\,, \tag{38}$$

which possess kinematical poles whenever $\theta_i=\xi_j$ and therefore need to be regularised[1]. Let A denote a set of one-particle excitations and A_1 and A_2 a partition of A. The scattering matrix arising from the commutations necessary to rewrite $|A\rangle$ as $|A_2A_1\rangle$ is denoted by S_{AA_1}, *i.e.*, $|A\rangle = S_{AA_1}\,|A_2A_1\rangle = S_{AA_2}\,|A_1A_2\rangle$. For example, if $|A\rangle = |\theta_1,\dots,\theta_5\rangle$ and $|A_1\rangle = |\theta_2,\theta_3\rangle$, then

$$|\theta_1,\dots,\theta_5\rangle = S_{AA_1}\,|\theta_1,\theta_4,\theta_5,\theta_2,\theta_3\rangle \quad \text{with} \quad S_{AA_1}=1. \tag{39}$$

If A and B denote two sets of one-particle excitations, the regularisation of the form factors (38) reads[1]

$$\langle A|\,\sigma^{\mathrm{z}}\,|B\rangle = \sum_{\substack{A=A_1\cup A_2\\ B=B_1\cup B_2}} d(B_2)\,S_{AA_1}\,S_{B_1B}\,\langle A_2|\,B_2\rangle\,\langle A_1+\mathrm{i}\,0|\,\sigma^{\mathrm{z}}\,|B_1\rangle \tag{40}$$

$$= \sum_{\substack{A=A_1\cup A_2\\ B=B_1\cup B_2}} S_{AA_2}\,S_{B_2B}\,\langle A_2|\,B_2\rangle\,\langle A_1-\mathrm{i}\,0|\,\sigma^{\mathrm{z}}\,|B_1\rangle\,, \tag{41}$$

where the sums are over all possible ways to break the sets $A=A_1\cup A_2$ and $B=B_1\cup B_2$ into subsets. The scalar products $\langle A_2|\,B_2\rangle$ are easily evaluated using (6). The factor $d(A)$ is present by virtue of the semi-locality of the spin operator with respect to the fundamental field and is given by

$$d(A) = (-1)^{n(A)}, \tag{42}$$

where $n(A)$ denotes the number of elements in A. As all rapidities in the remaining matrix elements are distinct, they can be evaluated using the crossing relations

$$\begin{aligned}\big\langle\theta_{i_1}\pm\mathrm{i}\,0,\dots,\theta_{i_p}\pm\mathrm{i}\,0\big|\,\sigma^{\mathrm{z}}\,\big|\xi_{j_1},\dots,\xi_{j_q}\big\rangle &= \langle 0|\,\sigma^{\mathrm{z}}\,\big|\theta_{i_1}+\mathrm{i}\,\pi\pm\mathrm{i}\,\eta_{i_1},\dots,\theta_{i_p}+\mathrm{i}\,\pi\pm\mathrm{i}\,\eta_{i_p},\xi_{j_1},\dots,\xi_{j_q}\big\rangle\\ &= f(\theta_{i_1}+\mathrm{i}\,\pi\pm\mathrm{i}\,\eta_{i_1},\dots,\theta_{i_p}+\mathrm{i}\,\pi\pm\mathrm{i}\,\eta_{i_p},\xi_{j_1},\dots,\xi_{j_q}).\end{aligned} \tag{43}$$

We stress that equations (40), (41) and (42) are valid only for the operator σ^{z} in the Ising model. For the disorder operator μ^{z} we have $l_{\mu^{\mathrm{z}}\mu^{\mathrm{z}}}=1$ and as a result the factor $d(B_2)$ in (40) needs to be dropped, whereas (41) remains unchanged. In other theories and in particular for operators with $l_{O\Psi}\neq\pm1$ (where Ψ denotes the fundamental field), additional phase factors related to the non-locality of the operators appear.

We have checked the validity of (40) and (41) using a finite-size regularisation for the Ising model (see below).

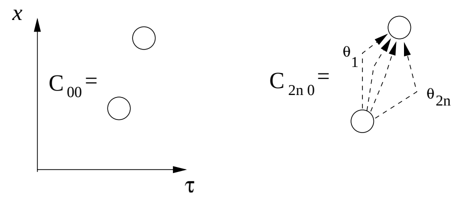

FIG. 7: Graphical representation of C_{00} and $C_{2n\,0}$. The circles represent the operators σ^z, where the upper one corresponds to the later time x. The particles created by the $A^\dagger(\theta)$'s are represented by the arrows. We order the arrows between the circles from left to right as $\theta_1, \ldots, \theta_{2n}$.

B. Spin-spin correlation function in the bulk system

The simplest terms in the expansion (35) are those with $m = 0$, *i.e.,* terms without boundary contributions. We recall the definition of the centre-of-mass coordinates $R = (x_1 + x_2)/2 \leq 0$ and $r = x_2 - x_1$. We find $C_{00}(\tau, r) = \sigma_0^2 \propto M^{1/4}$ as well as

$$C_{2n\,0}(\tau, r) = \frac{1}{(2n)!}\int_{-\infty}^{\infty} \frac{d\theta_1 \ldots d\theta_{2n}}{(2\pi)^{2n}} \left|f(\theta_1, \ldots, \theta_{2n})\right|^2 e^{-M|r|\sum_i \cosh\theta_i}\, e^{\mathrm{i}\,\mathrm{sgn}(r)\,M\tau\sum_i \sinh\theta_i}. \tag{44}$$

We note that the exponential factor $e^{-M|r|\sum_i \cosh\theta_i}$ ensures the convergence of the integrals. This is a general feature of the expressions we obtain. As we will see later, the results for the spectral function remain finite even in the limit $r \to 0$. In order to facilitate the comparison to the standard bulk results we now shift the contours of integration in (44) as follows. If we write $\theta = s + \mathrm{i}\,\varphi$, the exponential factors are given by

$$e^{-M|r|\cosh\theta}\, e^{\mathrm{i}\,\mathrm{sgn}(r)\,M\tau\sinh\theta} = e^{\mathrm{i}\,(\mathrm{sgn}(r)\,M\tau\cos\varphi - M|r|\sin\varphi)\sinh s}\, e^{-(\mathrm{sgn}(r)\,M\tau\sin\varphi + M|r|\cos\varphi)\cosh s}, \tag{45}$$

which vanishes exponentially for $s \to \pm\infty$ provided that

$$\mathrm{sgn}(r)\,M\tau\sin\varphi + M|r|\cos\varphi > 0. \tag{46}$$

Hence, in the first and third quadrants of the (τ, r)-plane, *i.e.,* when $\tau r > 0$, the contributions at $\mathfrak{Re}\,\theta_i = \pm\infty$ vanish as long as $0 \leq \varphi \leq \pi/2$ and we may shift the contour of integration $\theta_i \to \theta_i + \mathrm{i}\,\pi/2$ without changing the result. In the second and fourth quadrants ($\tau r < 0$) we may shift $\theta_i \to \theta_i - \mathrm{i}\,\pi/2$ instead. As the form factors have no poles in the strip $-\pi/2 \leq \mathfrak{Im}\,\theta_i \leq \pi/2$, we obtain (see also Appendix A)

$$C_{2n\,0}(\tau, r) = \frac{\sigma_0^2}{(2n)!}\int_{-\infty}^{\infty} \frac{d\theta_1 \ldots d\theta_{2n}}{(2\pi)^{2n}} \prod_{\substack{i,j=1\\ i<j}}^{2n} \tanh^2\frac{\theta_i - \theta_j}{2}\, e^{-\mathrm{i}\,M|r|\sum_i \sinh\theta_i}\, e^{-M|\tau|\sum_i \cosh\theta_i}. \tag{47}$$

In order to arrive a this expression we have replaced $\theta_i \to -\theta_i$ in the second and fourth quadrants. The result (47) is the well-known result for the spin-spin correlation function in the scaling limit of the Ising model[34]. In comparison to (44) we have effectively reversed the Euclidean rotation done in Sec. III, *i.e.,* we have interchanged τ and r back in order to interpret r as space and $\tau = \mathrm{i}\,t$ as Euclidean time.

If we shift the contours of integration by $\theta_i \to \theta_i \pm \mathrm{i}\,\arctan(\tau/r)$, (44) can be cast in the following form

$$C_{2n\,0}(\tau, r) = \frac{\sigma_0^2}{(2n)!}\int_{-\infty}^{\infty} \frac{d\theta_1 \ldots d\theta_{2n}}{(2\pi)^{2n}} \prod_{\substack{i,j=1\\ i<j}}^{2n} \tanh^2\frac{\theta_i - \theta_j}{2}\, e^{-M\sqrt{r^2+\tau^2}\sum_i \cosh\theta_i}. \tag{48}$$

In real space, $\tau = \mathrm{i}\,t$, we observe oscillating behaviour for time-like separations and damped behaviour for space-like separations.

C. First-order contributions in the boundary reflection matrix

In this subsection we calculate the leading contribution in the boundary reflection matrix K. The first, time-independent, term is given by

$$C_{02}(\tau, x_1, x_2) = -\mathrm{i}\,\sigma_0^2 \int_0^{\infty} \frac{d\xi}{2\pi}\, K(\xi)\, \tanh\xi\, e^{2M\max(x_1,x_2)\cosh\xi}. \tag{49}$$

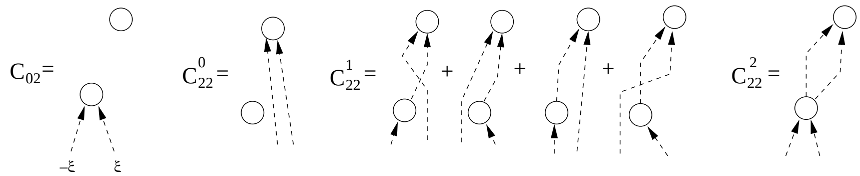

FIG. 8: Graphical representation of C_{02} and C_{22}. The arrows starting at the lower edge represent the particles $A^\dagger(-\xi)$ and $A^\dagger(\xi)$ coming from the boundary state. If these arrows pass the lower circle, which represents an operator $\sigma^{\rm z}$, it indicates that some of the internal particles created by the $A^\dagger(\theta_i)$'s have been contracted with the external lines, *i.e.,* terms like $\langle\theta_i|\,\xi\rangle = 2\pi\delta(\theta_i-\xi)$ appear in the corresponding formulas. The upper index denotes the number of lines connecting the two operators.

We note that C_{02} is cancelled in the connected correlation function (37). The first term containing a contribution of the boundary to the two-particle continuum is also the first term which contains matrix elements of the form (38),

$$C_{22}(\tau,x_1,x_2) = \frac{1}{2}\int_0^\infty \frac{d\xi}{2\pi}\int_{-\infty}^\infty \frac{d\theta_1 d\theta_2}{(2\pi)^2}\,K(\xi)\,f(\theta_1,\theta_2)\;\langle\theta_2,\theta_1|\,\sigma^{\rm z}\,|{-\xi},\xi\rangle \\ \cdot\, e^{2M\max(x_1,x_2)\cosh\xi}\,e^{-M|r|\sum_i\cosh\theta_i}\,e^{{\rm i}\,{\rm sgn}(r)\,M\tau\sum_i\sinh\theta_i}. \tag{50}$$

For the evaluation of the second matrix element we can use either (40) or (41), which give

$$\begin{aligned}\langle\theta_2,\theta_1|\,\sigma^{\rm z}\,|{-\xi},\xi\rangle \;=\;& (2\pi)^2\,\sigma_0\big(\delta(\theta_1+\xi)\,\delta(\theta_2-\xi)-\delta(\theta_1-\xi)\,\delta(\theta_2+\xi)\big)\\ &+2\pi\delta(\theta_1-\xi)\,f(\theta_2+{\rm i}\,\pi+{\rm i}\,\eta_2,-\xi)-2\pi\delta(\theta_1+\xi)\,f(\theta_2+{\rm i}\,\pi+{\rm i}\,\eta_2,\xi)\\ &-2\pi\delta(\theta_2-\xi)\,f(\theta_1+{\rm i}\,\pi+{\rm i}\,\eta_1,-\xi)+2\pi\delta(\theta_2+\xi)\,f(\theta_1+{\rm i}\,\pi+{\rm i}\,\eta_1,\xi)\\ &+f(\theta_2+{\rm i}\,\pi+{\rm i}\,\eta_2,\theta_1+{\rm i}\,\pi+{\rm i}\,\eta_1,-\xi,\xi) \qquad (51)\\ =\;& (2\pi)^2\,\sigma_0\big(\delta(\theta_1+\xi)\,\delta(\theta_2-\xi)-\delta(\theta_1-\xi)\,\delta(\theta_2+\xi)\big)\\ &-2\pi\delta(\theta_1-\xi)\,f(\theta_2+{\rm i}\,\pi-{\rm i}\,\eta_2,-\xi)+2\pi\delta(\theta_1+\xi)\,f(\theta_2+{\rm i}\,\pi-{\rm i}\,\eta_2,\xi)\\ &+2\pi\delta(\theta_2-\xi)\,f(\theta_1+{\rm i}\,\pi-{\rm i}\,\eta_1,-\xi)-2\pi\delta(\theta_2+\xi)\,f(\theta_1+{\rm i}\,\pi-{\rm i}\,\eta_1,\xi)\\ &+f(\theta_2+{\rm i}\,\pi-{\rm i}\,\eta_2,\theta_1+{\rm i}\,\pi-{\rm i}\,\eta_1,-\xi,\xi). \qquad (52)\end{aligned}$$

Both regularisations give altogether six contributions, which can be represented graphically as shown in Fig. 8. Either regularisation scheme gives rise to a contribution

$$C_{22}^0 = -{\rm i}\,\sigma_0^2\int_0^\infty \frac{d\xi}{2\pi}K(\xi)\,\tanh\xi\,e^{2M\min(x_1,x_2)\cosh\xi}, \tag{53}$$

where the additional upper index denotes the number of lines connecting the two operators. The contribution C_{22}^0 is similar to C_{02} and like the latter cancels in the connected correlation function.

The next term, C_{22}^1, actually consists of four contributions which can be combined using the relation $K(-\xi) = -K(\xi)$

$$C_{22}^1 = \mp\sigma_0^2\int_{-\infty}^\infty \frac{d\xi}{2\pi}\frac{d\theta}{2\pi}\,K(\xi)\,\tanh\frac{\xi-\theta}{2}\,\coth\frac{\xi+\theta\pm{\rm i}\,\eta}{2}\,e^{2MR\cosh\xi}\,e^{{\rm i}\;{\rm sgn}(r)\,M\tau\sinh\xi}\,e^{-M|r|\cosh\theta}\,e^{{\rm i}\;{\rm sgn}(r)\,M\tau\sinh\theta}. \tag{54}$$

For free boundary conditions the integral over ξ has to be understood as a principal value integration. Furthermore, the upper sign corresponds to the regularisation (51) and the lower sign to (52). The difference between the two is compensated by analogous differences in the contribution C_{22}^2 discussed below. In order to take the limit $\eta\to 0$ we shift the contour of the θ-integration in the same way as we did for the bulk terms $C_{2n\,0}$. As the exponential factors containing θ in (54) equal those in (44), we obtain again the condition (46) for the vanishing of the integrals at $\mathfrak{Re}\,\theta\to\pm\infty$. Hence, if the space-time coordinates (τ,r) lie in the first or third quadrants, $\tau r>0$, we have to shift $\theta\to\theta+{\rm i}\,\pi/2$. In order to avoid the appearance of extra terms arising from the residues of the coth in (54) we choose the regularisation (51), for which the pole in the θ-plane is located at $\theta=-\xi-{\rm i}\,\eta$, *i.e.,* below the real axis. By the same reasoning we choose the regularisation (52) if $\tau r<0$. Doing so we obtain

$$C_{22}^1 \;=\; \mp\sigma_0^2\int_{-\infty}^\infty \frac{d\xi}{2\pi}\frac{d\theta}{2\pi}\,K(\xi)\,\tanh\frac{\xi-\theta\mp{\rm i}\,\pi/2}{2}\,\coth\frac{\xi+\theta\pm{\rm i}\,\pi/2\pm{\rm i}\,\eta}{2} \\ \cdot e^{2MR\cosh\xi}\,e^{{\rm i}\,{\rm sgn}(r)\,M\tau\sinh\xi}\,e^{-{\rm i}\,{\rm sgn}(\tau)\,Mr\sinh\theta}\,e^{-M|\tau|\cosh\theta} \tag{55}$$

$$= \pm\sigma_0^2 \int_{-\infty}^{\infty} \frac{d\xi}{2\pi}\frac{d\theta}{2\pi}\, K(\xi)\, \frac{\cosh\theta \pm \mathrm{i}\,\sinh\xi}{\cosh\theta \mp \mathrm{i}\,\sinh\xi}\, e^{2MR\cosh\xi}\, e^{-\mathrm{i}\,\mathrm{sgn}(\tau)\, Mr\sinh\theta}\, e^{\mathrm{i}\,\mathrm{sgn}(r)\, M\tau\sinh\xi}\, e^{-M|\tau|\cosh\theta}, \qquad (56)$$

where we have taken the limit $\eta \to 0$. In the second and fourth quadrants of the (τ, r)-plane we now change variables $\xi \to -\xi$ and $\theta \to -\theta$, which in particular changes $e^{\mathrm{i}\,\mathrm{sgn}(r)\, M\tau\sinh\xi}$ to $e^{\mathrm{i}\, M|\tau|\sinh\xi}$. Hence, as $R < 0$ we can shift $\xi \to \xi + \mathrm{i}\,\pi/2$ and obtain

$$C_{22}^1(\tau, x_1, x_2) = -\sigma_0^2 \int_{-\infty}^{\infty} \frac{d\xi}{2\pi}\frac{d\theta}{2\pi}\, K\big(\xi + \mathrm{i}\,\tfrac{\pi}{2}\big)\, \frac{\cosh\xi - \cosh\theta}{\cosh\xi + \cosh\theta}\, e^{\mathrm{i}\,2MR\sinh\xi}\, e^{-\mathrm{i}\, M|r|\sinh\theta}\, e^{-M|\tau|(\cosh\theta+\cosh\xi)} \qquad (57)$$

$$+\Theta(h_c - h)\; 2\sigma_0^2\, \cot v\, \tan\frac{v}{2}\, e^{2MR\cos v}\, e^{-M|\tau|\sin v} \cdot \int_{-\infty}^{\infty} \frac{d\theta}{2\pi}\, \frac{\cosh\theta - \sin v}{\cosh\theta + \sin v}\, e^{-\mathrm{i}\, M|r|\sinh\theta}\, e^{-M|\tau|\cosh\theta}. \qquad (58)$$

Here (58) originates from the pole of $K(\xi)$ at $\xi = \mathrm{i}\, v$ (we recall $\kappa = 1 - h^2/2M = \cos v$) and is present if $h < h_c$. In particular, for free boundary conditions we obtain $\sigma_0^2\, e^{2MR}\, K_0\big(M\sqrt{r^2+\tau^2}\big)/\pi$. Explicit expressions for $K\big(\xi + \mathrm{i}\,\frac{\pi}{2}\big)$ are given in (A9)–(A11). We will see below that (57) yields an oscillating contribution to the spectral function.

Going through the same steps as above, we find that the final term in the regularisation (51) or (52) respectively can be cast in the form

$$C_{22}^2 = -\mathrm{i}\,\frac{\sigma_0^2}{2} \int_0^{\infty} \frac{d\xi}{2\pi} \int_{-\infty}^{\infty} \frac{d\theta_1 d\theta_2}{(2\pi)^2}\, K(\xi)\, \tanh\xi \prod_{i=1}^{2} \coth\frac{\xi - \theta_i \mp \mathrm{i}\,\eta_i}{2}\, \coth\frac{\xi + \theta_i \pm \mathrm{i}\,\eta_i}{2} \cdot \tanh^2\frac{\theta_1 - \theta_2}{2}\, e^{2M\max(x_1,x_2)\cosh\xi}\, e^{-M|r|\sum_i\cosh\theta_i}\, e^{\mathrm{i}\,\mathrm{sgn}(r)\, M\tau\sum_i\sinh\theta_i} \qquad (59)$$

$$= -\mathrm{i}\,\frac{\sigma_0^2}{2} \int_0^{\infty} \frac{d\xi}{2\pi} \int_{-\infty}^{\infty} \frac{d\theta_1 d\theta_2}{(2\pi)^2}\, K(\xi)\, \tanh\xi\, \tanh^2\frac{\theta_1 - \theta_2}{2} \prod_{i=1}^{2} \frac{\cosh\xi \pm \mathrm{i}\,\sinh\theta_i}{\cosh\xi \mp \mathrm{i}\,\sinh\theta_i} \cdot e^{2M\max(x_1,x_2)\cosh\xi}\, e^{\mp\mathrm{i}\, M|r|\sum_i\sinh\theta_i}\, e^{-M|\tau|\sum_i\cosh\theta_i}. \qquad (60)$$

Here we have shifted $\theta_i \to \theta_i + \mathrm{i}\,\pi/2$ for $\tau r > 0$ and $\theta_i \to \theta_i - \mathrm{i}\,\pi/2$ for $\tau r < 0$, respectively, and taken the limits $\eta_i \to 0$. Finally, we can substitute $\theta_i \to -\theta_i$ if $\tau r < 0$.

Puting everything together, the result for C_{22} reads

$$\begin{aligned} C_{22}(\tau, x_1, x_2) = &-\mathrm{i}\,\sigma_0^2 \int_0^{\infty} \frac{d\xi}{2\pi} K(\xi)\, \tanh\xi\, e^{2M\min(x_1,x_2)\cosh\xi} \\ &-\sigma_0^2 \int_{-\infty}^{\infty} \frac{d\xi}{2\pi}\frac{d\theta}{2\pi}\, K\big(\xi + \mathrm{i}\,\tfrac{\pi}{2}\big)\, \frac{\cosh\xi - \cosh\theta}{\cosh\xi + \cosh\theta}\, e^{\mathrm{i}\,2MR\sinh\xi}\, e^{-\mathrm{i}\, M|r|\sinh\theta}\, e^{-M|\tau|(\cosh\theta+\cosh\xi)} \\ &-\mathrm{i}\,\frac{\sigma_0^2}{2} \int_0^{\infty} \frac{d\xi}{2\pi} \int_{-\infty}^{\infty} \frac{d\theta_1 d\theta_2}{(2\pi)^2} K(\xi)\, \tanh\xi\, \tanh^2\frac{\theta_1 - \theta_2}{2} \prod_{i=1}^{2} \frac{\cosh\xi + \mathrm{i}\,\sinh\theta_i}{\cosh\xi - \mathrm{i}\,\sinh\theta_i} \\ &\qquad \cdot e^{2M\max(x_1,x_2)\cosh\xi}\, e^{-\mathrm{i}\, M|r|\sum_i\sinh\theta_i}\, e^{-M|\tau|\sum_i\cosh\theta_i} \\ &+\Theta(h_c - h)\; 2\sigma_0^2\, \cot v\, \tan\frac{v}{2}\, e^{2MR\cos v}\, e^{-M|\tau|\sin v} \\ &\qquad \cdot \int_{-\infty}^{\infty} \frac{d\theta}{2\pi}\, \frac{\cosh\theta - \sin v}{\cosh\theta + \sin v}\, e^{-\mathrm{i}\, M|r|\sinh\theta}\, e^{-M|\tau|\cosh\theta}. \end{aligned} \qquad (61)$$

The last term is the contribution due to the boundary bound state.

If one uses regularisation (52) instead of (51) in the first or third quadrants of the (τ, r)-plane, one has to keep track of the poles of the coth in (56) as well as (60) in the strip $0 \le \mathfrak{Im}\,\theta \le \pi/2$. A straightforward calculation shows that the contributions of these residues cancel out and that one finds again the result (61). In order to check the regularisation scheme, we have performed a finite-size regularisation of C_{22}. The results are presented in Appendix B and equal (61) within a relative error of less than 10^{-4}.

In order to study the light-cone effect in more detail, let us reconsider C_{22}^1 in the region $h_c < h$. We start with (56) and shift the contours of integration $\theta \to \theta - \mathrm{i}\,\theta_0$ and $\xi \to \xi + \mathrm{i}\,\xi_0$, where $\theta_0 = \arctan\big(|r|/|\tau|\big)$ and $\xi_0 = \arctan\big(|\tau|/2|R|\big)$.

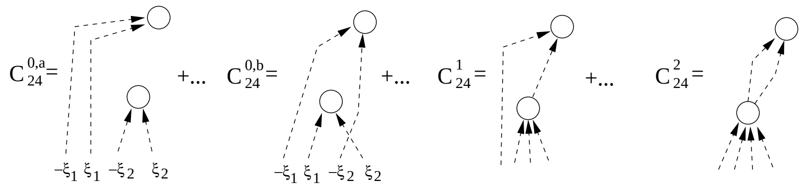

FIG. 9: Graphical representation of C_{24}. The first term, $C_{24}^{0,\mathrm{a}}$, is completely disconnected. In the second term, $C_{24}^{0,\mathrm{b}}$, the contractions of the particles in the intermediate state and the particles in the boundary state are crossed and the operators become intertwined.

This yields

$$C_{22}^{1}=\sigma_0^2\int_{-\infty}^{\infty}\frac{d\xi}{2\pi}\frac{d\theta}{2\pi}\,K\big(\xi+\mathrm{i}\,\xi_0\big)\,\frac{\cosh\!\big(\theta-\mathrm{i}\,\theta_0\big)+\mathrm{i}\,\sinh\!\big(\xi+\mathrm{i}\,\xi_0\big)}{\cosh\!\big(\theta-\mathrm{i}\,\theta_0\big)-\mathrm{i}\,\sinh\!\big(\xi+\mathrm{i}\,\xi_0\big)}\,e^{-M\sqrt{r^2+\tau^2}\cosh\theta}\,e^{-M\sqrt{4R^2+\tau^2}\cosh\xi}. \tag{62}$$

We observe the same qualitative features as for the Green's function (31), *i.e.,* oscillating behaviour for $r^2<t^2$ as well as $4R^2<t^2$.

In Appendix C we calculate further contributions to first order in the boundary reflection matrix, but with higher numbers of particles in the intermediate state. In particular, we show that $C_{42}^2(\tau,x_1,x_2)=C_{22}^2(\tau,x_2,x_1)$.

D. Second-order contributions in the boundary reflection matrix

In this subsection we calculate the leading contributions to second order in the boundary reflection matrix K. The first term of this kind, C_{04}, drops out in the calculation of the connected correlation function. The next term is C_{24}. For simplicity, we will restrict ourselves to the first quadrant in the (τ,r)-plane, *i.e.,* $\tau>0$ and $r=x_2-x_1>0$:

$$\begin{aligned}C_{24}(\tau,x_1,x_2)=\frac{1}{4}\int_0^{\infty}\frac{d\xi_1 d\xi_2}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{d\theta_1 d\theta_2}{(2\pi)^2}\,K(\xi_1)\,K(\xi_2)\,e^{2Mx_2\sum_i\cosh\xi_i}\,e^{-Mr\sum_i\cosh\theta_i}\\ \cdot\,e^{\mathrm{i}\,M\tau\sum_i\sinh\theta_i}\,f(\theta_1,\theta_2)\,\langle\theta_2,\theta_1|\,\sigma^{\mathrm{z}}\,|{-\xi_1},\xi_1,-\xi_2,\xi_2\rangle\,.\end{aligned} \tag{63}$$

Using (40) the second matrix element can be written as

$$\begin{aligned}\langle\theta_2,\theta_1|\,\sigma^{\mathrm{z}}\,|{-\xi_1},\xi_1,-\xi_2,\xi_2\rangle =\ &\langle\theta_2,\theta_1|-\xi_1,\xi_1\rangle\,f(-\xi_2,\xi_2)+\langle\theta_2,\theta_1|-\xi_2,\xi_2\rangle\,f(-\xi_1,\xi_1)\\ &-\langle\theta_2,\theta_1|\,\xi_1,\xi_2\rangle\,f(-\xi_1,-\xi_2)+\langle\theta_2,\theta_1|-\xi_1,\xi_2\rangle\,f(\xi_1,-\xi_2)\\ &+\langle\theta_2,\theta_1|\,\xi_1,-\xi_2\rangle\,f(-\xi_1,\xi_2)-\langle\theta_2,\theta_1|-\xi_1,-\xi_2\rangle\,f(\xi_1,\xi_2)\\ &-\langle\theta_2|\,\xi_2\rangle\,f(\theta_1+\mathrm{i}\,\pi+\mathrm{i}\,\eta_1,-\xi_1,\xi_1,-\xi_2)+\langle\theta_2|-\xi_2\rangle\,f(\theta_1+\mathrm{i}\,\pi+\mathrm{i}\,\eta_1,-\xi_1,\xi_1,\xi_2)\\ &-\langle\theta_2|\,\xi_1\rangle\,f(\theta_1+\mathrm{i}\,\pi+\mathrm{i}\,\eta_1,-\xi_1,-\xi_2,\xi_2)+\langle\theta_2|-\xi_1\rangle\,f(\theta_1+\mathrm{i}\,\pi+\mathrm{i}\,\eta_1,\xi_1,-\xi_2,\xi_2)\\ &+\langle\theta_1|\,\xi_2\rangle\,f(\theta_2+\mathrm{i}\,\pi+\mathrm{i}\,\eta_2,-\xi_1,\xi_1,-\xi_2)-\langle\theta_1|-\xi_2\rangle\,f(\theta_2+\mathrm{i}\,\pi+\mathrm{i}\,\eta_2,-\xi_1,\xi_1,\xi_2)\\ &+\langle\theta_1|\,\xi_1\rangle\,f(\theta_2+\mathrm{i}\,\pi+\mathrm{i}\,\eta_2,-\xi_1,-\xi_2,\xi_2)-\langle\theta_1|-\xi_1\rangle\,f(\theta_2+\mathrm{i}\,\pi+\mathrm{i}\,\eta_2,\xi_1,-\xi_2,\xi_2)\\ &+f(\theta_2+\mathrm{i}\,\pi+\mathrm{i}\,\eta_2,\theta_1+\mathrm{i}\,\pi+\mathrm{i}\,\eta_1,-\xi_1,\xi_1,-\xi_2,\xi_2).\end{aligned} \tag{64}$$

A graphical representation for the contribution (63) using the decomposition (64) is shown in Fig. 9. We observe that there exist two different terms in which the operators are not directly connected. The first one, $C_{24}^{0,\mathrm{a}}$, is completely disconnected and hence does not contribute to the connected correlation function (37). The second one, $C_{24}^{0,\mathrm{b}}$, is obtained from the second and third lines of (64). Although there is no line connecting the two operators, the contractions of the particles in the intermediate state and the particles in the boundary state are crossed and the operators become intertwined. Hence, this term does contribute to the connected correlation function. It is explicitly given by (for free boundary conditions the integrals have to be understood as principal value integrations)

$$C_{24}^{0,\mathrm{b}}(\tau,x_1,x_2)\ =\ -\frac{\sigma_0^2}{2}\int_{-\infty}^{\infty}\frac{d\xi_1 d\xi_2}{(2\pi)^2}\,K(\xi_1)\,K(\xi_2)\,\tanh^2\frac{\xi_1-\xi_2}{2}\,e^{2MR\sum_i\cosh\xi_i}\,e^{\mathrm{i}\,M\tau\sum_i\sinh\xi_i}$$

$$= -\frac{\sigma_0^2}{2}\int_{-\infty}^{\infty}\frac{d\xi_1 d\xi_2}{(2\pi)^2}\,K\big(\xi_1+\mathrm{i}\,\tfrac{\pi}{2}\big)\,K\big(\xi_2+\mathrm{i}\,\tfrac{\pi}{2}\big)\tanh^2\frac{\xi_1-\xi_2}{2}\,e^{\mathrm{i}\,2MR\sum_i\sinh\xi_i}\,e^{-M\tau\sum_i\cosh\xi_i} \tag{65}$$

$$-\Theta(h_{\rm c}-h)\;2\sigma_0^2\,\cot v\,\tan\frac{v}{2}\,e^{2MR\cos v}\,e^{-M|\tau|\sin v}$$
$$\cdot\int_{-\infty}^{\infty}\frac{d\xi}{2\pi}\,K\big(\xi+\mathrm{i}\,\tfrac{\pi}{2}\big)\,\tanh^2\frac{\xi+\mathrm{i}\,(\pi/2-v)}{2}\,e^{\mathrm{i}\,2MR\sinh\xi}\,e^{-M\tau\cosh\xi}. \tag{66}$$

We note that the result for free boundary conditions is obtained as the limit $v\to 0$. In the derivation of (66) we used that the function had no pole at $\xi_1=\xi_2=\mathrm{i}\,v$. Contribution (65) can be rewritten as

$$-\frac{\sigma_0^2}{2}\int_{-\infty}^{\infty}\frac{d\xi_1 d\xi_2}{(2\pi)^2}\,K(\xi_1+\mathrm{i}\,\xi_0)\,K(\xi_2+\mathrm{i}\,\xi_0)\,\tanh^2\frac{\xi_1-\xi_2}{2}\,e^{-M\sqrt{4R^2+\tau^2}\sum_i\cosh\xi_i}, \tag{67}$$

which again shows oscillating behaviour for $4R^2<t^2$ in real time. The other terms appearing in (64) yield sub-leading contributions to the correlation function and are determined in Appendix C.

The calculation of higher-order terms in the boundary reflection matrix can be performed along the same lines as above (see Appendix C). Although no principal problems appear, the calculations become soon rather tedious. The expectation is, however, that the first few orders in K will give accurate results even rather close to the boundary. We show below that this is indeed the case for the local spectral function, where the terms calculated above are found to be sufficient for $M|R|\gtrsim 0.2$. The calculation of terms in the spectral representation with a higher number of particles in the intermediate state is also discussed in Appendix C. The corresponding terms in the local spectral function are found to be negligible compared to the two-particle contributions. The analogous behaviour for the bulk Ising model is well documented[7,38,45], and it was argued by Cardy and Mussardo that this behaviour is a general feature of form factor expansions in integrable field theories[46].

E. Limiting case

Let us consider the behaviour of the two-point function in the limit $R\to-\infty$ in more detail. We have $\min(x_1,x_2)\to-\infty$, but $\max(x_1,x_2)$ may remain close to the boundary. Hence, in addition to oscillating terms like C_{22}^1 there are contributions of terms like C_{22}^2, C_{42}^4 and C_{24}^2. These terms diverge in the limit $\max(x_1,x_2)\to 0$, *i.e.,* the series expansion (35) ceases to converge. In the case $\min(x_1,x_2)=0$ one has to deal with an operator located at the boundary. The properties of such boundary operators and their counterparts in the bulk are, in general, very different. For example, σ^z has the boundary scaling dimension 1/2, while its scaling dimensions in the bulk are $(1/16,1/16)$[23]. The treatment of boundary operators in the framework of integrable field theories and form factor expansions has been put forward recently by Bajnok, Palla and Takács[47].

The scaling behaviour of the two-point function in the case of free boundary conditions was derived for the lattice Ising model by Bariev[36]. He obtained the following result in the region $\sqrt{r^2+\tau^2}>2R\gg 1/M$,

$$\langle 0_{\rm B}|\,\delta\sigma^z(\tau,x_1)\,\delta\sigma^z(0,x_2)\,|0_{\rm B}\rangle\sim\frac{1}{(r^2+\tau^2)^{1/4}}\,e^{2MR}\,e^{-M\sqrt{r^2+\tau^2}}. \tag{68}$$

In the field theoretical calculation the leading term of the left-hand side equals C_{22}^1, which is according to (57) and (58) given by

$$C_{22}^1(\tau,x_1,x_2) = \frac{\sigma_0^2}{\pi}\,e^{2MR}\,K_0\big(M\sqrt{r^2+\tau^2}\big) \tag{69}$$
$$-\sigma_0^2\int_{-\infty}^{\infty}\frac{d\xi}{2\pi}\frac{d\theta}{2\pi}\,K\big(\xi+\mathrm{i}\,\tfrac{\pi}{2}\big)\,\frac{\cosh\xi-\cosh\theta}{\cosh\xi+\cosh\theta}\,e^{\mathrm{i}\,2MR\sinh\xi}\,e^{-\mathrm{i}\,M|r|\sinh\theta}\,e^{-M|\tau|(\cosh\theta+\cosh\xi)}. \tag{70}$$

In the region $\sqrt{r^2+\tau^2}\gg 1/M$ the asymptotic behaviour[44] of the first term is given by (68). On the other hand we can evaluate the second term in the stationary phase approximation, which yields

$$-\frac{\sigma_0^2}{2\pi M}\,K\big(\xi_0+\mathrm{i}\,\tfrac{\pi}{2}\big)\,\frac{\sqrt{r^2+\tau^2}-\sqrt{4R^2+\tau^2}}{\sqrt{r^2+\tau^2}+\sqrt{4R^2+\tau^2}}\,\frac{e^{-M\sqrt{4R^2+\tau^2}}}{(4R^2+\tau^2)^{1/4}}\,\frac{e^{-M\sqrt{r^2+\tau^2}}}{(r^2+\tau^2)^{1/4}},\quad \xi_0=\mathrm{i}\,\arctan\left(\frac{2R}{|\tau|}\right). \tag{71}$$

The condition $\sqrt{r^2+\tau^2}>2R\gg 1/M$ translates into $\tau^2>4x_1x_2>0$, where the second inequality follows from the fact that the result for C_{22}^1 is valid only if both operators are located away from the boundary. The condition $\tau^2>0$,

however, implies that (71) is exponentially suppressed compared to (69). Hence the asymptotic behaviour of C_{22}^1 is given by (69) and equals the result derived by Bariev in the lattice Ising model.

We would also like to comment on the conformal limit $M \to 0$ of the Ising model. The boundary conditions compatible with conformal symmetry are the fixed and free ones. Using the method of mirror images[22,41] one can derive the two-point function of σ^{z}. If one considers the limit $x_2 \to 0$ while keeping x_1 and τ fixed, one finds

$$\langle 0_{\mathrm{B}}|\,\sigma^{\mathrm{z}}(1/M,1/M)\,\sigma^{\mathrm{z}}(0,x_2)\,|0_{\mathrm{B}}\rangle \sim \begin{cases} x_2^{3/8}, & \text{free boundary conditions,} \\ x_2^{-1/8}, & \text{fixed boundary conditions.} \end{cases} \tag{72}$$

Although we cannot apply the expansion (35) in this limit, we mention that this result is in agreement with the leading correction to the correlation function derived above. For $\min(x_1,x_2)$ and τ kept fixed this correction is given by (49), which is negative in the case of free boundary conditions and hence suggesting the vanishing of the correlation function, but positive for fixed boundary conditions, thus supporting a diverging behaviour. However, in order to study the vicinity of the conformally invariant point more accurately one has use different methods like the truncated conformal space approach[24,25,26].

Finally, we mention that it is not possible to resum higher-order contributions in K by a geometric series, which can be achieved for finite-temperature correlation functions in the Ising model[48] and the non-linear sigma model[10].

VII. SPECTRAL FUNCTION

In order to gain further physical insight in the spin correlations it is useful to calculate the corresponding spectral function. The positive frequency part is given by

$$C(\omega,x_1,x_2) = \int_0^\infty d\tau\, e^{\mathrm{i}\,\bar{\omega}\tau}\, C(\tau,x_1,x_2)\Big|_{\bar{\omega}\to -\mathrm{i}\,\omega+\delta}, \tag{73}$$

where the analytic continuation of the frequencies is $\bar{\omega} = -\mathrm{i}\,\omega + \delta$.

There are essentially four different types of terms in the series (37), which we will discuss below separately. First, we recover the known results for the bulk from all terms with $m=0$. Second, there exist terms which essentially yield corrections to the two- and four-particle continua already present in the bulk. The leading corrections of this kind are C_{22}^2 and C_{42}^2 as well as C_{42}^4 and C_{62}^4. Third, we find oscillatory behaviour, which is present even deep in the bulk ($M|R| \gg 1$). Here the leading contributions are C_{22}^1 and $C_{24}^{0,\mathrm{b}}$. Fourth, we discuss the contributions of the boundary bound state for boundary magnetic fields $h < h_{\mathrm{c}}$. The numerical evaluation of the integrals appearing in the spectral functions was performed using the VEGAS routine for Monte Carlo integration[49].

A. Bulk Result

As we have mentioned before, the bulk results for the n-particle level are given by $C_{2n\,0}$. The full correlation function in the bulk is

$$C_{\mathrm{bulk}}(\tau,r) = \sum_{n=0}^\infty \frac{\sigma_0^2}{(2n)!}\int_{-\infty}^\infty \frac{d\theta_1\ldots d\theta_{2n}}{(2\pi)^{2n}} \prod_{\substack{i,j=1\\ i<j}}^{2n} \tanh^2\frac{\theta_i-\theta_j}{2}\, e^{-\mathrm{i}\,M|r|\sum_i \sinh\theta_i}\, e^{-M|\tau|\sum_i\cosh\theta_i}. \tag{74}$$

If we perform the Fourier transformation $\tau \to \omega$ and calculate the spectral function $S_{\mathrm{bulk}} = \mathfrak{Im}\, C_{\mathrm{bulk}}$, we find for the two-particle continuum

$$S_{\mathrm{bulk}}^2(\omega,r) = \frac{\sigma_0^2\pi}{2}\int_{-\infty}^\infty \frac{d\theta_1 d\theta_2}{(2\pi)^2}\tanh^2\frac{\theta_1-\theta_2}{2}\,\cos\Big(Mr\sum\nolimits_i \sinh\theta_i\Big)\,\delta\Big(\omega - M\sum\nolimits_i\cosh\theta_i\Big), \tag{75}$$

where we have restricted ourselves to positive frequencies $\omega > 0$. Resolving the δ-functions gives

$$S_{\mathrm{bulk}}^2(\omega,r) = 2\sigma_0^2 \int_0^{\mathrm{Arcosh}\left(\frac{\omega}{2M}\right)} \frac{d\theta}{2\pi}\,\frac{\tanh^2\theta}{\sqrt{\omega^2-4M^2\cosh^2\theta}}\,\cos\Big(|r|\sqrt{\omega^2-4M^2\cosh^2\theta}\Big). \tag{76}$$

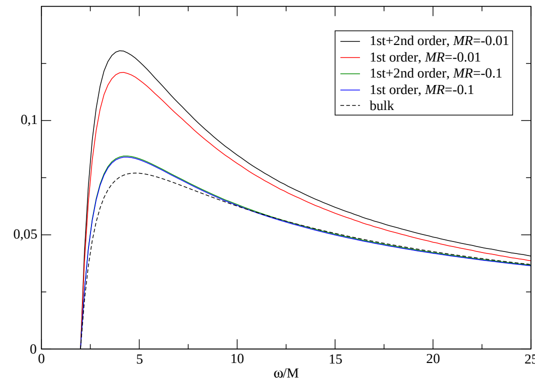

FIG. 10: Total two-particle contribution to the local spectral density (79) for fixed boundary conditions (in units of σ_0^2/M). The bulk result is obtained for $MR \to -\infty$. We observe that for $MR = -0.1$ the second order contribution is already very small.

We stress that (76) vanishes very slowly for large separation r at fixed frequency ω. A similar behaviour is found for the two-point function of the disorder operator μ^z, whose leading contribution is given by the one-particle peak (see Sec.VIII). The four-particle continuum can be readily calculated from the corresponding term in (74). One finds that the ratio $S^4_{\text{bulk}}/S^2_{\text{bulk}}$ is $\sim 1/150$ at $\omega/M = 25$ and smaller for lower energies. This suppression of the higher-order terms in the series (74) is a well-known feature in the Ising model and other massive theories[7,38,45,46,50,51].

B. Two- and four-particle continua

We next turn to terms in the spectral representation (35), which have a direct counterpart in the bulk (74).

The two-particle continuum to first order in the boundary reflection matrix is given by C^2_{22} and C^2_{42}, *i.e.*, terms with two lines connecting the two operators. Performing the Fourier transform, analytically continuing and taking the imaginary part we find after a straightforward calculation using the formulas given in Appendix A

$$\begin{aligned}
S^2_{1K}(\omega,R,r) = \sigma_0^2 \int_0^\infty \frac{d\xi}{2\pi} \int_{-\theta'}^{\theta'} \frac{d\theta}{2\pi} \, & \frac{\hat{K}(\xi)\,\tanh\xi}{\sqrt{(\omega - M\cosh\theta)^2 - M^2}} \, \frac{e^{2MR\cosh\xi}}{\cosh^2\xi + \sinh^2\theta} \, \frac{\tanh^2\frac{\theta-\tilde{\theta}}{2}}{\cosh^2\xi + \sinh^2\tilde{\theta}} \\
\cdot \Big\{ & \Big[\big(\cosh^2\xi - \sinh^2\theta\big)\big(\cosh^2\xi - \sinh^2\tilde{\theta}\big) - 4\cosh^2\xi \sinh\theta \sinh\tilde{\theta} \Big] \\
& \cdot \cos\big(M|r|(\sinh\theta + \sinh\tilde{\theta})\big)\, \cosh\big(M|r|\cosh\xi\big) \\
& + 2\cosh\xi \Big[\big(\cosh^2\xi - \sinh^2\tilde{\theta}\big)\sinh\theta + (\cosh^2\xi - \sinh^2\theta)\sinh\tilde{\theta} \Big] \\
& \cdot \sin\big(M|r|(\sinh\theta + \sinh\tilde{\theta})\big)\, \sinh\big(M|r|\cosh\xi\big) \Big\},
\end{aligned} \tag{77}$$

where

$$\tilde{\theta} = \tilde{\theta}(\omega,\theta) = \text{Arcosh}\left(\frac{\omega}{M} - \cosh\theta\right) \quad \text{and} \quad \theta' = \text{Arcosh}\left(\frac{\omega}{M} - 1\right). \tag{78}$$

We observe that the integrand is exponentially suppressed for large distances from the boundary by the factor $e^{2MR\cosh\xi}$. Furthermore, we note that (77) is particle-hole symmetric. The two-particle continuum to second order in K, which we denote by S^2_{2K}, is obtained from C^2_{24}, $C^{2,\text{a}}_{44}$ and C^2_{64}. Its explicit form is derived in Appendix D 1.

The total two-particle contribution to the local spectral density is given by summation of all terms with two lines connecting the two operators. In Fig. 10 we have plotted the truncation at the second order in K,

$$S^2_{\text{tot}}(\omega,R,r=0) = S^2_{\text{bulk}}(\omega,r=0) + S^2_{1K}(\omega,R,r=0) + S^2_{2K}(\omega,R,r=0) + \ldots, \tag{79}$$

for fixed boundary conditions. The results show that the effects of the boundary can be treated in perturbation theory for $M|R| \gtrsim 0.2$. This is also supported by the study of the ratio of S^2_{2K} to S^2_{1K} at their respective peaks, which is plotted in Fig. 11. For example, for fixed boundary conditions and $MR = -0.1$ the latter is about 5%.

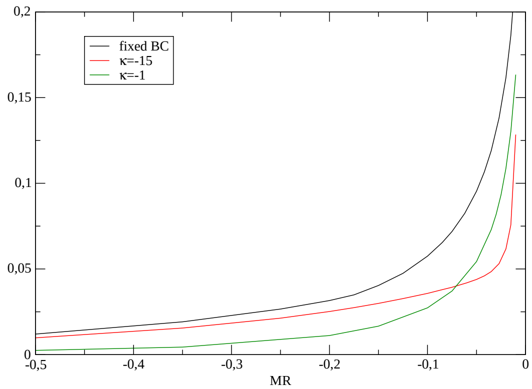

FIG. 11: Ratio of the first and second order (in the boundary reflection matrix) contributions to the two-particle continua S^2_{2K}/S^2_{1K} at their respective peaks for different distances from the boundary.

Starting from C^4_{42} and C^4_{62} we have calculated the four-particle contribution to the spectral function S^4_{1K} to first order in K. The explicit result is given in Appendix D 1. Comparison with S^2_{1K} shows that the four-particle contribution is negligible in the low-energy region $\omega \lesssim 25M$. This indicates that the fast rate of convergence in the number of particles in the intermediate state, which is a well-known fact for the Ising model in the bulk[7,38,45,46], is also present in the system with boundary.

C. Oscillating contributions

The corrections to the bulk we have studied so far are negligible for $MR \ll -1$. In this section we discuss oscillating terms which affect the spectral function also deep in the bulk. The leading term of this kind is given by (57), we will denote it by $C^{1,\text{osc}}_{22}$. The corresponding contribution to the spectral function reads

$$S^{1,\text{osc}}_{22}(\omega, R, r=0) = \frac{2\sigma_0^2}{\omega} \int\limits_0^{\text{Arcosh}\left(\frac{\omega}{M}-1\right)} \frac{d\xi}{2\pi} \, \frac{\omega - 2M\cosh\xi}{\sqrt{(\omega - M\cosh\xi)^2 - M^2}} \cdot \Big[\mathfrak{Re}K\big(\xi + \mathrm{i}\,\tfrac{\pi}{2}\big)\,\cos\big(2MR\sinh\xi\big) - \mathfrak{Im}K\big(\xi + \mathrm{i}\,\tfrac{\pi}{2}\big)\,\sin\big(2MR\sinh\xi\big)\Big], \tag{80}$$

where $K\big(\xi + \mathrm{i}\,\frac{\pi}{2}\big)$ is given by (A9)–(A11). For simplicity we have restricted ourselves to $r = 0$. We stress that (80) is not exponentially suppressed for large distances from the boundary. In fact, the integral (80) possesses a square-root singularity at the upper limit $\xi = \text{Arcosh}\left(\frac{\omega}{M} - 1\right)$. Expanding the non-singular part of the integrand about this value yields

$$S^{1,\text{osc}}_{22}(\omega, R, r=0) \sim \frac{\cos\big(\Delta R + \varphi\big)}{\sqrt{M|R|}}, \quad \Delta = 2\sqrt{\omega^2 - 2M\omega}, \tag{81}$$

where φ depends on the boundary conditions.

The next term we want to analyse is $C^{0,\text{b},\text{osc}}_{24}$. The corresponding spectral function is found to be

$$S^{0,\text{b},\text{osc}}_{24}(\omega, R, r) = -\frac{\sigma_0^2}{2M} \int_{-\xi'}^{\xi'} \frac{d\xi}{2\pi} \, \frac{1}{\sqrt{(\omega - M\cosh\xi)^2 - M^2}} \, \tanh^2 \frac{\xi - \tilde{\xi}}{2} \cdot \Big\{ \Big[\mathfrak{Re}K\big(\xi + \mathrm{i}\,\tfrac{\pi}{2}\big)\,\mathfrak{Re}K\big(\tilde{\xi} + \mathrm{i}\,\tfrac{\pi}{2}\big) - \mathfrak{Im}K\big(\xi + \mathrm{i}\,\tfrac{\pi}{2}\big)\,\mathfrak{Im}K\big(\tilde{\xi} + \mathrm{i}\,\tfrac{\pi}{2}\big)\Big] \cos\big(2MR(\sinh\xi + \sinh\tilde{\xi})\big) - 2\,\mathfrak{Re}K\big(\xi + \mathrm{i}\,\tfrac{\pi}{2}\big)\,\mathfrak{Im}K\big(\tilde{\xi} + \mathrm{i}\,\tfrac{\pi}{2}\big)\,\sin\big(2MR(\sinh\xi + \sinh\tilde{\xi})\big)\Big\}, \tag{82}$$

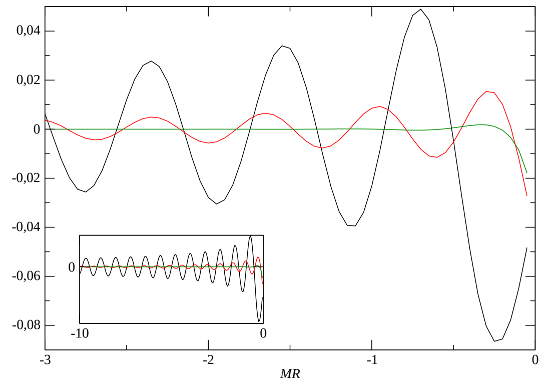

FIG. 12: Oscillating terms $S_{22}^{1,\mathrm{osc}}$ (black line) and $S_{24}^{0,\mathrm{b,osc}}$ (red line) as well as $S_{24}^{1,\mathrm{osc}} + S_{44}^{1,\mathrm{osc}}$ (green line) for fixed boundary conditions, $\omega = 5M$ and $r = 0$ (in units of σ_0^2/M). We stress that $S_{22}^{1,\mathrm{osc}}$ and $S_{24}^{0,\mathrm{b,osc}}$ are present deep in the bulk, whereas $S_{24}^{1,\mathrm{osc}} + S_{44}^{1,\mathrm{osc}}$ is strongly suppressed for large $M|R|$.

where $\tilde{\xi}$ and ξ' are given by

$$\tilde{\xi} = \tilde{\xi}(\omega, \xi) = \mathrm{Arcosh}\left(\frac{\omega}{M} - \cosh\xi\right), \quad \xi' = \mathrm{Arcosh}\left(\frac{\omega}{M} - 1\right). \tag{83}$$

We note that (82) is independent of r. Numerical evaluation of the integral in the range $MR \leq -20$ and $\omega \approx 5M$ shows that

$$S_{24}^{0,\mathrm{b,osc}}(\omega, R, r = 0) \sim \frac{\cos\bigl(\Delta' R + \varphi'\bigr)}{(M|R|)^{3/2}}, \quad \Delta' > \Delta. \tag{84}$$

The larger oscillation frequency Δ' is due to the additional term $\sinh\tilde{\xi} \geq 0$ in the sine and cosine in (82). Although (84) falls off faster than (81), $S_{24}^{0,\mathrm{b,osc}}$ should not be interpreted as a higher-order correction to $S_{22}^{1,\mathrm{osc}}$, as both terms are present deep in the bulk. The analysis of the corresponding contributions in the presence of a boundary bound state (see below) rather suggests that $S_{24}^{0,\mathrm{b,osc}}$ represents an independent oscillating contribution of order K; and hence that the total oscillating contribution to first order in the boundary reflection matrix is given by

$$S_{1K}^{\mathrm{osc}}(\omega, R, r) = S_{22}^{1,\mathrm{osc}}(\omega, R, r) + S_{24}^{0,\mathrm{b,osc}}(\omega, R, r). \tag{85}$$

The higher-order corrections in K to (80) and (82) are in fact obtained starting from $C_{24}^{1,\mathrm{osc}} + C_{44}^{1,\mathrm{osc}}$ and $C_{26}^{0,\mathrm{osc}} + C_{46}^{0,\mathrm{osc}}$, respectively. The corresponding spectral function S_{2K}^{osc} is evaluated in Appendix D 2. We note that the appearing integrands are exponentially suppressed for large distances from the boundary.

We have plotted (80) and (82) as well as (D5) in Fig. 12 for fixed boundary conditions, fixed energy and varying distance from the boundary. The first two are present in the bulk ($MR \sim -10$), whereas the third one is strongly suppressed for large $M|R|$. This underlines the interpretation of (D5) as first correction to (80). Finally, the correction to (80) with three particles in the intermediate state is given by $C_{42}^{3,\mathrm{osc}}$. It is found to be negligible for all values of MR and energies $\omega \leq 25M$. The same result holds for $C_{44}^{2,\mathrm{b,osc}}$, the first correction to (82) with more internal lines. This again underlines that the series (35) is rapidly convergent as $n \to \infty$.

D. Contributions from the boundary bound state

In the previous section we have investigated the oscillatory behaviour of the two-point function. This behaviour was caused by the terms C_{22}^1, $C_{24}^{0,\mathrm{b}}$, $C_{24}^1 + C_{44}^1$ and $C_{26}^0 + C_{46}^0$. In Sec. VI we have seen that in the range $h < h_{\mathrm{c}}$ all these terms possess additional contributions due to the existence of the boundary bound state, which we will now discuss in detail.

We first consider (58), which we will denote by $C_{22}^{1,\mathrm{bbs}}$. The corresponding contribution to the spectral function can be cast in the form

$$S_{22}^{1,\mathrm{bbs}}(\omega, R, r) = \frac{2\sigma_0^2\,\cot v\,\tan\frac{v}{2}}{\sqrt{(\omega - M\sin v)^2 - M^2}}\,\frac{\omega - 2M\sin v}{\omega}\,\cos\Bigl(r\sqrt{(\omega - M\sin v)^2 - M^2}\Bigr)\,e^{2MR\cos v}, \tag{86}$$

and is seen to be exponentially suppressed in MR. We recall that the parameter v is defined as $\kappa = 1-h^2/2M = \cos v$. In the limit $v \to \pi/2$ the suppression of $S_{22}^{1,\mathrm{bbs}}$ becomes less and less effective. In this limit the boundary bound state becomes weakly bound and its effective size diverges[27].

Similarly we obtain

$$
\begin{aligned}
S_{24}^{0,\mathrm{b,bbs}}(\omega,R,r) =& -\frac{2M^2\sigma_0^2\,\cot v\,\tan\frac{v}{2}}{\sqrt{(\omega-M\sin v)^2-M^2}}\,\frac{e^{2MR\cos v}}{\omega^2}\\
&\cdot\Bigg\{\left[\frac{1}{2}\,\mathfrak{Re}K\big(\hat\xi+\mathrm{i}\,\tfrac{\pi}{2}\big)\,\big(\cosh 2\hat\xi-\cos 2v-2\big)-2\,\mathfrak{Im}K\big(\hat\xi+\mathrm{i}\,\tfrac{\pi}{2}\big)\,\sinh\hat\xi\,\cos v\right]\cos\big(2MR\sinh\hat\xi\big)\\
&-\left[2\,\mathfrak{Re}K\big(\hat\xi+\mathrm{i}\,\tfrac{\pi}{2}\big)\,\sinh\hat\xi\,\cos v+\frac{1}{2}\,\mathfrak{Im}K\big(\hat\xi+\mathrm{i}\,\tfrac{\pi}{2}\big)\,\big(\cosh 2\hat\xi-\cos 2v-2\big)\right]\sin\big(2MR\sinh\hat\xi\big)\Bigg\},
\end{aligned}
\tag{87}
$$

where

$$
\hat\xi = \hat\xi(\omega,v) = \mathrm{Arcosh}\left(\frac{\omega}{M}-\sin v\right). \tag{88}
$$

The contribution $S_{24}^{0,\mathrm{b,bbs}}$ is independent of r and oscillates in R with the frequency-dependent wave number $2\sqrt{(\omega-M\sin v)^2-M^2}$. In particular, for large ω this wave number approaches $2(\omega-M\sin v)$. Furthermore, (87) is exponentially suppressed for $R\to-\infty$.

Interestingly, although both (86) and (87) have threshold singularities at $\omega = M+M\sin v$, these singularities cancel each other. This shows that we have to interpret their sum as the first-order contribution of the boundary bound state to the spectral function, *i.e.,* we have

$$
S_{1K}^{\mathrm{bbs}}(\omega,R,r) = S_{22}^{1,\mathrm{bbs}}(\omega,R,r)+S_{24}^{0,\mathrm{b,bbs}}(\omega,R,r). \tag{89}
$$

This indicates as well that the similar sum (85) should be viewed as the oscillating contribution to first order in K. We observe that S_{1K}^{bbs} has a gap $M+M\sin v$ which equals the energy of the boundary bound state plus one additional particle. On the other hand the spectral function vanishes at $\omega = M\sin v$. Combining these two observations we conclude that

$$
\langle 1_{\mathrm{B}}|\,\sigma^{\mathrm{z}}\,|0_{\mathrm{B}}\rangle = 0 \quad\text{but}\quad \langle 1_{\mathrm{B}},\theta|\,\sigma^{\mathrm{z}}\,|0_{\mathrm{B}}\rangle \neq 0, \tag{90}
$$

where we have used the notations in the original picture with the boundary located in space. The vanishing of the spectral function for $\omega\to M+M\sin v$ means that the rapidity-dependence of the matrix elements of σ^{z} overcomes the van-Hove type singularities in the one-particle density of states at $\theta=0$.

The result for free boundary conditions is obtained in the limit $v\to 0$. In this case (89) simplifies to

$$
S_{1K,\mathrm{free}}^{\mathrm{bbs}}(\omega,R,r) = \frac{\sigma_0^2\,e^{2MR}}{\sqrt{\omega^2-M^2}}\left[\cos\big(r\sqrt{\omega^2-M^2}\big)-\frac{M}{\omega}\left(\cos\big(2R\sqrt{\omega^2-M^2}\big)+\frac{\sqrt{\omega^2-M^2}}{M}\sin\big(2R\sqrt{\omega^2-M^2}\big)\right)\right] \tag{91}
$$

and the spectral gap equals M.

The second-order contribution in K of the boundary bound state to the spectral function is given by

$$
S_{2K}^{\mathrm{bbs}}(\omega,R,r) = S_{24}^{1,\mathrm{bbs}}(\omega,R,r)+S_{44}^{1,\mathrm{bbs}}(\omega,R,r)+S_{26}^{0,\mathrm{bbs}}(\omega,R,r)+S_{46}^{0,\mathrm{bbs}}(\omega,R,r). \tag{92}
$$

Explicit expressions for the various contributions are given in Appendix D 3. We find that all integrands appearing in S_{2K}^{bbs} are exponentially small for large $M|R|$. Furthermore, S_{2K}^{bbs} remains finite at the lower threshold $\omega\to M+M\sin v$, which is ensured by the cancellation of all singularities in (92). We expect similar cancellations of singularities to occur at all orders. It is also straightforward to calculate the first correction to (89) with more internal lines, which is given by $C_{42}^{3,\mathrm{bbs}}+C_{44}^{2,\mathrm{b,bbs}}$. We find this to be negligible for all values of R and energies $\omega\leq 25M$.

E. Results for the local spectral function

In the preceeding sections we have discussed individual terms in the expansion of the spectral function. We are now in a position to combine the various contributions and present results for the local spectral function of Ising spins up

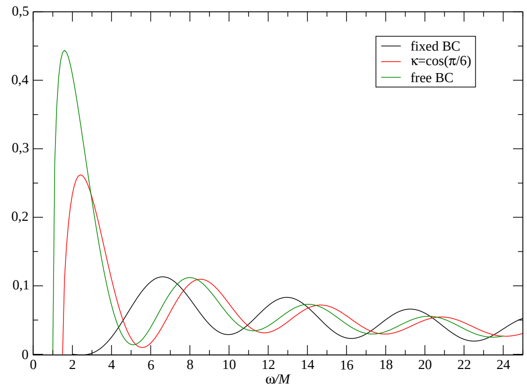

FIG. 13: Local spectral function (93) for $MR = -0.5$ and fixed and free boundary conditions as well as a boundary magnetic field corresponding to $v = \pi/6$ (in units of σ_0^2/M). The spectral function possesses a gap of $2M$ for fixed boundary conditions. For boundary magnetic fields $h < h_c$ the gap is smaller than $2M$ and goes to M in the limit of free boundary conditions.

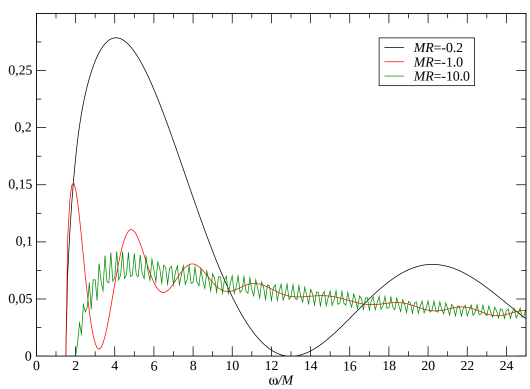

FIG. 14: Local spectral function (93) for a boundary magnetic field corresponding to $v = \pi/6$ and different distances from the boundary (in units of σ_0^2/M).

to second order in the boundary K-matrix. If we truncate the expansion at the two-particle level as well, the local spectral function truncated is given by

$$
\begin{aligned}
S^{\rm lsf}(\omega,R) &= S^{\rm lsf}_{\rm bulk}(\omega,R) + S^{\rm lsf}_{1K}(\omega,R) + S^{\rm lsf}_{2K}(\omega,R) + \ldots && (93)\\
S^{\rm lsf}_{\rm bulk}(\omega,R) &= S^{2}_{\rm bulk}(\omega,R,r=0) + \ldots && (94)\\
S^{\rm lsf}_{1K}(\omega,R) &= S^{2}_{1K}(\omega,R,r=0) + S^{\rm osc}_{1K}(\omega,R,r=0) + S^{\rm bbs}_{1K}(\omega,R,r=0) + \ldots && (95)\\
S^{\rm lsf}_{2K}(\omega,R) &= S^{2}_{2K}(\omega,R,r=0) + S^{\rm osc}_{2K}(\omega,R,r=0) + S^{\rm bbs}_{2K}(\omega,R,r=0) + \ldots && (96)
\end{aligned}
$$

where the terms originating in the boundary bound state are only present if $h < h_c$. The dots in (93) represent higher orders in K, whereas the dots in (94)–(96) stand for terms with more than two particles in the intermediate state. As we have shown above, the truncation at second order in K is sufficient for $MR \lesssim -0.2$. We have further argued that the truncation at the two-particle level gives a very good accuracy for energies $\omega \leq 25M$.

We have plotted (93) in Fig. 13 for fixed and free boundary conditions as well as a boundary magnetic field corresponding to $\kappa = \cos(\pi/6)$. For fixed boundary conditions the local spectral function has a gap $2M$. The same behaviour of is well-known from the system in the bulk. For sufficient small values of the boundary magnetic field, however, the gap is given by $E_1 + M$, where E_1 is the energy of the boundary bound state. This shows that the matrix element of σ^z between the vacuum state in the presence of the boundary and states containing the boundary bound state and one additional particle is non-zero. In the limit of free boundary conditions E_1 tends to zero and the gap approaches the single-particle gap M. Furthermore, we observe oscillating behaviour in ω for all three boundary conditions.

In Fig. 14 we plot (93) for a boundary magnetic field corresponding to $\kappa = \cos(\pi/6)$. Close to the boundary the spectral weight is concentrated at low energies and in particular at energies $\omega \leq 2M$. The weight below $2M$ is due

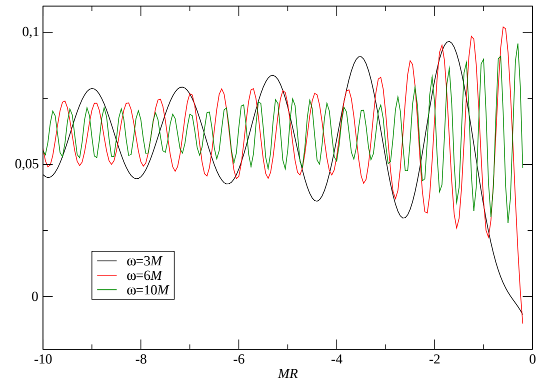

FIG. 15: Local spectral function (93) for fixed boundary conditions as function of the distance from the boundary (in units of σ_0^2/M).

to the creation of the boundary bound state and one additional particle. The term responsible for this behaviour is (89). Both (86) and (87) are exponentially suppressed with increasing distance from the boundary. Hence the spectral weight in the region $0 \le \omega \le 2M$ is very small, as can be seen in Fig. 14 for $MR = -10$. This strong suppression may complicate an experimental detection of this effect of the boundary bound state on the local spectral function. Furthermore, the spectral function oscillates in ω with a frequency depending on the distance from the boundary. For large $|R|$ these oscillations become very rapid, which may cause difficulties in detecting this effect as well.

Finally, we plot the local spectral function as a function of the distance from the boundary in Fig. 15. It is clear from this plot that the oscillations are only algebraically decaying in $|R|$.

In summary, the local spectral function (93) exhibits two new features in the presence of a boundary. First, the spectral function shows oscillatory behaviour both as a function of ω with a frequency depending on the distance from the boundary, and as a function of R with an energy dependent frequency. Second, for values of the boundary magnetic field below the critical value $h_{\rm c} = \sqrt{2M}$ we have found spectral weight in the interval $M \le \omega \le 2M$, *i.e.,* within the gap of the bulk Ising model. This spectral weight is due to the existence of a boundary bound state, and hence strongly suppressed with increasing distance from the boundary.

Any quasi one-dimensional material described by the quantum Ising model needs to be thought of as an ensemble of finite length chains. If the average chain length is large, a model in terms of semi-infinite chains should constitute a good starting point and the results obtained here should be applicable. In particular, it should be possible to detect the midgap states in inelastic neutron scattering experiments. On the other hand, the oscillatory behaviour of dynamical spin correlations is likely to average out.

VIII. CORRELATION FUNCTION OF THE DISORDER FIELD

In this section we briefly discuss the two-point function of the disorder field $\mu^{\rm z}$. Up to first order in K we find

$$\begin{aligned}\langle 0_{\rm B}|\, \mathcal{T}_\tau\, \mu^{\rm z}(\tau,x_1)\, \mu^{\rm z}(0,x_2)\, |0_{\rm B}\rangle &= \langle 0|\, \mathcal{T}_x\, \mu^{\rm z}(\tau,x_1)\, \mu^{\rm z}(0,x_2)\, |{\rm B}\rangle \\ &= \langle 0|\, \mu^{\rm z}(\tau,x_1)\, \mu^{\rm z}(0,x_2)\, |0\rangle + \int_0^\infty \frac{d\xi}{2\pi}\, K(\xi)\, \langle 0|\, \mu^{\rm z}(\tau,x_1)\, \mu^{\rm z}(0,x_2)\, |{-\xi},\xi\rangle + \ldots, \end{aligned} \tag{97}$$

where we have assumed that $x_1 < x_2$. The first term equals the result in the bulk. The single-particle contribution to it is

$$\sigma_0^2 \int_{-\infty}^{\infty} \frac{d\theta}{2\pi}\, e^{-Mr\cosh\theta}\, e^{{\rm i}\,M\tau\sinh\theta} = \sigma_0^2 \int_{-\infty}^{\infty} \frac{d\theta}{2\pi}\, e^{-M\sqrt{r^2+\tau^2}\cosh\theta} = \frac{\sigma_0^2}{\pi}\, K_0\big(M\sqrt{r^2+\tau^2}\big), \tag{98}$$

where we have used (A7). After analytic continuation, $\tau = {\rm i}\,t$, we observe typical light-cone behaviour. Fourier transforming we find that the single-particle contribution to the bulk part of the spectral function is given by

$$S^{\mu^{\rm z}}_{\rm bulk}(\omega > M, r > 0) = \frac{\sigma_0^2}{\sqrt{\omega^2 - M^2}}\, \cos\Big(r\sqrt{\omega^2 - M^2}\Big). \tag{99}$$

We note that the spectral function shows undamped oscillations as function of r.

If we assume $\tau > 0$, the leading contribution due to the presence of the boundary can be calculated using (40)

$$\langle\theta|\,\mu^z\,|{-\xi},\xi\rangle = \langle\theta \pm \mathrm{i}\,0|\,\mu^z\,|{-\xi},\xi\rangle + 2\pi\sigma_0\,\delta(\theta-\xi) - 2\pi\sigma_0\,\delta(\theta+\xi). \tag{100}$$

The connected piece merely yields a correction to the one-particle spectral function (99) which vanishes for large distances from the boundary. However, the disconnected pieces give

$$\sigma_0^2 \int_{-\infty}^{\infty} \frac{d\xi}{2\pi}\, K(\xi)\, e^{2MR\cosh\theta}\, e^{\mathrm{i}\,M\tau\sinh\theta} \;=\; \sigma_0^2 \int_{-\infty}^{\infty} \frac{d\xi}{2\pi}\, K(\xi+\mathrm{i}\,\xi_0)\, e^{-M\sqrt{4R^2+\tau^2}\cosh\xi} \tag{101}$$

$$+\Theta(h_{\mathrm{c}}-h)\;\Theta(\xi_0-v)\;2\sigma_0^2\,\cot v\,\tan\frac{v}{2}\,e^{2MR\cos v}\,e^{-M\tau\sin v}, \tag{102}$$

where $\xi_0 = \arctan(\tau/2|R|)$. Note that (101) and (102) are independent of r. If we perform the analytic continuation $\tau = \mathrm{i}\,t$ in (101), we find an exponentially damped behaviour for $4R^2 > t^2$, and an oscillating behaviour for $4R^2 < t^2$. The contribution to the spectral function corresponding to (101) is given by

$$S_{1\mathrm{K}}^{\mu^z,\mathrm{osc}} = \frac{\sigma_0^2}{\sqrt{\omega^2-M^2}}\Big[\mathfrak{Re}\,K\big(\mathrm{Arcosh}\tfrac{\omega}{M}+\mathrm{i}\,\tfrac{\pi}{2}\big)\,\cos\!\big(2\sqrt{\omega^2-M^2}\,R\big) - \mathfrak{Im}\,K\big(\mathrm{Arcosh}\tfrac{\omega}{M}+\mathrm{i}\,\tfrac{\pi}{2}\big)\,\sin\!\big(2\sqrt{\omega^2-M^2}\,R\big)\Big]. \tag{103}$$

We observe that (103) is not damped for large distances $|R|$. This is due to the dissipationless propagation of a single particle to the boundary and back, and a likewise dissipationless reflection off the boundary. On the other hand, (102) represents the contribution of the boundary bound state. It is seen to give rise to a δ-peak contribution to the spectral function at the energy $\omega = M\sin v$, which implies that $\langle 1_{\mathrm{B}}|\,\mu^z\,|0_{\mathrm{B}}\rangle \neq 0$. Higher order contributions to the two-point function of disorder operators can be calculated by the same method we employed above for the spin correlations.

IX. CONCLUSIONS

Our main result is the calculation of the dynamical spin-spin correlations in the Ising field theory with a boundary. We have derived an expansion in powers of the boundary reflection matrix K and shown that it converges rapidly even in the case where both operators in the two-point function are fairly close to the boundary. We have also demonstrated that like in the bulk case higher-order terms in the expansion in the number of particles in the intermediate state are negligible at low energies. The most notable effect of the boundary is that at sufficiently late times the spin-spin correlations show oscillatory behaviour arbitrarily far away from the boundary. As is well known, for small values of the boundary magnetic field a boundary bound state exists. This bound state leads to a contribution to the spectral function within the gap of the bulk Ising chain. Similar features are also found for the Green's functions of the Majorana fermions and the two-point function of the disorder field.

We have seen that the expansion in powers of the boundary K-matrix breaks down close to the boundary. In order to access this regime, other methods are necessary. One possible approach is the truncated conformal space approach, which has already been applied successfully to the analogous problem for one-point functions[26]. In would be interesting to generalise these results to the case of two-point functions and in this way obtain accurate expressions for the two-point function for all values of MR.

The results obtained in the present work have applications not only to the quantum Ising model itself. It is well-known that both anisotropic spin-1 Heisenberg chains[52] and the weak-coupling two-leg spin-1/2 ladder[53] can be described in terms of three and four Ising models respectively. Correlation functions of the staggered components of the spin operators in these models are represented by products of two-point functions of the spin and disorder fields of the quantum Ising chain. Using the results obtained in the present work one may calculate dynamical correlation functions for these systems.

Finally, having established that the expansion in powers of the boundary K-matrix converges even relatively close to the boundary, one may apply the method used here to other systems. In a forthcoming work we apply the method used in the present work to the calculation of the local tunneling density of states in a one-dimensional charge density wave state[54].

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APPENDIX A: USEFUL FORMULAS

We summarise some formulas, which were frequently used in the derivation of the $C_{2n\,2m}$'s.

$$\sinh(x \pm \mathrm{i}\,\pi/2) \;=\; \pm\mathrm{i}\,\cosh x, \tag{A1}$$

$$\cosh(x \pm \mathrm{i}\,\pi/2) \;=\; \pm\mathrm{i}\,\sinh x, \tag{A2}$$

$$\tanh\frac{x-y}{2}\,\tanh\frac{x+y}{2} \;=\; \frac{\cosh x - \cosh y}{\cosh x + \cosh y}, \tag{A3}$$

$$\tanh\frac{x-y\mp\mathrm{i}\,\pi/2}{2}\,\tanh\frac{x+y\pm\mathrm{i}\,\pi/2}{2} \;=\; \frac{\cosh x \mp \mathrm{i}\,\sinh y}{\cosh x \pm \mathrm{i}\,\sinh y} \tag{A4}$$

$$\tanh\frac{x-y\mp\mathrm{i}\,\pi/2}{2}\,\coth\frac{x+y\pm\mathrm{i}\,\pi/2}{2} \;=\; -\frac{\cosh y \pm \mathrm{i}\,\sinh x}{\cosh y \mp \mathrm{i}\,\sinh x}, \tag{A5}$$

$$\coth\frac{x-y\mp\mathrm{i}\,\pi/2}{2}\,\coth\frac{x+y\pm\mathrm{i}\,\pi/2}{2} \;=\; \frac{\cosh x \pm \mathrm{i}\,\sinh y}{\cosh x \mp \mathrm{i}\,\sinh y}. \tag{A6}$$

If $x, y \in \mathbb{R}$, $y > 0$, $f(\theta)$ is analytic in $-\arctan(x/y) \le \mathfrak{Im}\,\theta \le \arctan(x/y)$ and at most exponentially growing for $\theta \to \pm\infty$, then

$$\int_{-\infty}^{\infty}\frac{d\theta}{2\pi}\,f(\theta)\,e^{\mathrm{i}\,Mx\sinh\theta}\,e^{-My\cosh\theta} = \int_{-\infty}^{\infty}\frac{d\theta}{2\pi}\,f(\theta+\mathrm{i}\,\theta_0)\,e^{-M\sqrt{x^2+y^2}\cosh\theta}, \tag{A7}$$

where $\theta_0 = \arctan(x/y)$. In particular, if $\alpha \in \mathrm{R}$ and $f(\theta) = e^{\alpha\theta}$ then.

$$\int_{-\infty}^{\infty}\frac{d\theta}{2\pi}\,e^{\alpha\theta}\,e^{\mathrm{i}\,Mx\sinh\theta}\,e^{-My\cosh\theta} = \left(\frac{\mathrm{i}\,y-x}{\mathrm{i}\,y+x}\right)^{\alpha/2}\int_{-\infty}^{\infty}\frac{d\theta}{2\pi}\,e^{\alpha\theta}\,e^{-M\sqrt{x^2+y^2}\cosh\theta}. \tag{A8}$$

If f possesses poles in the strip $-\arctan(x/y) \le \mathfrak{Im}\,\theta \le \arctan(x/y)$ additional contributions appear on the right-hand side of (A7).

The explicit form of the boundary reflection matrix K at $\xi + \mathrm{i}\,\pi/2$ is

$$\begin{aligned}K\big(\xi+\mathrm{i}\,\tfrac{\pi}{2}\big) =& \frac{1}{1+\tanh^2\xi/2}\,\frac{1}{\kappa^2+\sinh^2\xi}\Bigg[\left(\tanh^2\frac{\xi}{2}-1\right)\left(\kappa^2-\sinh^2\xi\right) - 4\kappa\tanh\frac{\xi}{2}\,\sinh\xi \\ &+2\mathrm{i}\left(\tanh\frac{\xi}{2}\left(\kappa^2-\sinh^2\xi\right)+\kappa\sinh\xi\left(\tanh^2\frac{\xi}{2}-1\right)\right)\Bigg].\end{aligned} \tag{A9}$$

For the special cases of free and fixed boundary conditions this simplifies to

$$K_{\text{free}}\big(\xi+\mathrm{i}\,\tfrac{\pi}{2}\big) \;=\; \frac{1}{1+\coth^2\xi/2}\left[1-\coth^2\frac{\xi}{2}-2\mathrm{i}\,\coth\frac{\xi}{2}\right], \tag{A10}$$

$$K_{\text{fixed}}\big(\xi+\mathrm{i}\,\tfrac{\pi}{2}\big) \;=\; -\frac{1}{1+\tanh^2\xi/2}\left[1-\tanh^2\frac{\xi}{2}-2\mathrm{i}\,\tanh\frac{\xi}{2}\right]. \tag{A11}$$

We note that the real part is even under $\xi \to -\xi$ whereas the imaginary part is odd.

In the calculation of the spectral functions we use that if $\omega \in \mathbb{R}$, $f(\theta_1,\theta_2) = -f(-\theta_1,-\theta_2)$ and f sufficiently well behaved, then

$$\int_{-\infty}^{\infty}d\theta_1\,\mathcal{P}\!\int_{-\infty}^{\infty}d\theta_2\,\frac{f(\theta_1,\theta_2)}{\omega-\sum_i\cosh\theta_i} \;=\; 0, \tag{A12}$$

$$\int_{-\infty}^{\infty}d\theta_1 d\theta_2\,f(\theta_1,\theta_2)\,\delta\Big(\omega-\sum\nolimits_i\cosh\theta_i\Big) \;=\; 0. \tag{A13}$$

Furthermore, we use

$$\mathfrak{Re}\prod_{i=1}^{2}\frac{\cosh\xi+\mathrm{i}\,\sinh\theta_i}{\cosh\xi-\mathrm{i}\,\sinh\theta_i} = \frac{\prod_i^2(\cosh^2\xi-\sinh^2\theta_i)-4\cosh^2\xi\,\prod_i^2\sinh\theta_i}{\prod_i^2(\cosh^2\xi+\sinh^2\theta_i)}, \tag{A14}$$

$$\mathfrak{Im}\prod_{i=1}^{2}\frac{\cosh\xi+\mathrm{i}\,\sinh\theta_i}{\cosh\xi-\mathrm{i}\,\sinh\theta_i} = 2\cosh\xi\,\frac{\sum_i^2\sinh\theta_i\prod_{j\neq i}^2(\cosh^2\xi-\sinh^2\theta_j)}{\prod_i^2(\cosh^2\xi+\sinh^2\theta_i)}, \tag{A15}$$

and similar formulas for the higher-order terms. Note that (A14) is even under simultaneous reflection $\theta_i \to -\theta_i$, while (A15) is odd.

APPENDIX B: FINITE-SIZE REGULARISATION OF C_{22}

In order to check the infinite volume regularisation scheme for form factors involving two multiparticle states, we evaluate C_{22} in the finite system $0 \leq \tau \leq L$. A similar analysis was performed recently for the correlation functions in the Ising model at finite temperatures[55]. The form factors for σ^{z} in the finite system are[56]

$$_{\mathrm{NS}}\langle k_1,\ldots,k_m|\,\sigma^{\mathrm{z}}\,|n_1,\ldots,n_n\rangle_{\mathrm{R}} = S(L)\prod_{i=1}^{m}\tilde{g}(\xi_{k_i})\prod_{j=1}^{n}g(\theta_{n_j})F_{m,n}(\xi_{k_1},\ldots,\xi_{k_m}|\theta_{n_1},\ldots,\theta_{n_n}), \tag{B1}$$

where the rapidities in the Neveu–Schwarz (NS) and Ramond (R) sector are given by

$$\mathrm{NS}: \quad ML\sinh\xi_k = 2\pi k,\ k\in\mathbb{Z}+\tfrac{1}{2}, \tag{B2}$$
$$\mathrm{R}: \quad ML\sinh\theta_n = 2\pi n,\ n\in\mathbb{Z}. \tag{B3}$$

The function $F_{m,n}$ in (B1) is the infinite-volume form factor

$$F_{m,n}(\xi_1,\ldots,\xi_m|\theta_1,\ldots,\theta_n) = \mathrm{i}^{\lfloor(m+n)/2\rfloor}\sigma_0\prod_{\substack{i,j=1\\i<j}}^{m}\tanh\frac{\xi_i-\xi_j}{2}\prod_{\substack{i,j=1\\i<j}}^{n}\tanh\frac{\theta_i-\theta_j}{2}\prod_{i=1}^{m}\prod_{j=1}^{n}\coth\frac{\xi_i-\theta_j}{2}, \tag{B4}$$

where the symbol $\lfloor\ \rfloor$ denotes the floor function, *i.e.*, $\lfloor x\rfloor$ is the largest integer $l \leq x$, and the constant as well as the leg factors are

$$S(L) = 1+\mathcal{O}(e^{-L}), \quad g(\theta)=\tilde{g}(\theta)=\frac{1}{\sqrt{ML\cosh\theta}}+\mathcal{O}(e^{-L}). \tag{B5}$$

We note that due to the fact that σ^{z} connects the NS and R sectors of the Hilbert space, the rapidities ξ_i and θ_j cannot coincide and therefore no singularities occur in the finite volume.

In terms of these form factors the finite volume regularisation of C_{22} is given by (we assume $x_1 < x_2$ for simplicity)

$$C_{22} = \frac{1}{2}\sum_{\substack{k\in\mathrm{NS}\\k>0}}\sum_{n_1,n_2\in\mathrm{R}}K(\xi_k)\,{}_{\mathrm{NS}}\langle 0|\,\sigma^{\mathrm{z}}(\tau,x_1)\,|n_1,n_2\rangle_{\mathrm{R}}\ {}_{\mathrm{R}}\langle n_2,n_1|\,\sigma^{\mathrm{z}}(0,x_2)\,|{-k},k\rangle_{\mathrm{NS}} \tag{B6}$$

$$= -\mathrm{i}\,\frac{\sigma_0^2}{2}S(L)^2\sum_{\substack{k\in\mathrm{NS}\\k>0}}\sum_{n_1,n_2\in\mathrm{R}}K(\xi_k)\ e^{2Mx_2\cosh\xi_k}\ e^{-Mr\sum_i\cosh\theta_{n_i}}\ e^{\mathrm{i}\,M\tau\sum_i\sinh\theta_{n_i}}$$

$$\cdot\tilde{g}^2(\xi_k)\ \tanh\xi_k\ \tanh^2\frac{\theta_{n_1}-\theta_{n_2}}{2}\prod_{i=1}^{2}g^2(\theta_{n_i})\ \coth\frac{\xi_k-\theta_{n_i}}{2}\coth\frac{\xi_k+\theta_{n_i}}{2}. \tag{B7}$$

We have evaluated (B7) and (61) numerically for several values of x_1, x_2 and $0.2 \leq \tau \leq 1$ as well as different values of the boundary magnetic field. The two expressions coincide within a relative error of less than 10^{-4}. For the evaluation we used $L = 50$ and $N = 350$ as cut-off for the momenta, *i.e.*, $0 < k < N$ and $-N \leq n_i \leq N$. The result does not change if we use Ramond states as the outer states in (B6) instead of Neveu–Schwarz ones.

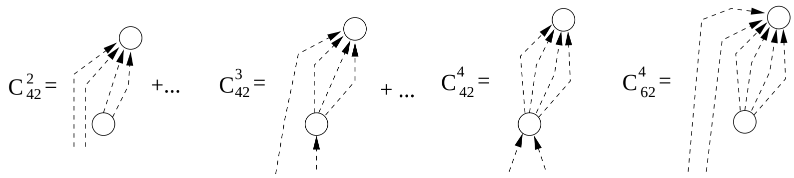

FIG. 16: Graphical representation of C_{42} and C^4_{62}. The dots represent diagrams with the same number of contractions.

APPENDIX C: HIGHER-ORDER CORRECTIONS TO (35)

In this appendix we calculate several higher-order terms in the expansion (36), which can be compared with the leading contributions calculated in Sec. VI. In order to simplify the notations, we will restrict ourselves to $\tau > 0$ and $r > 0$. The correlation function for general τ and r can be obtained as in Sec. VI.

1. Calculation of C_{42}

The next term in the series (35) is given by

$$C_{42}(\tau, x_1, x_2) = \frac{1}{4!}\int_0^\infty \frac{d\xi}{2\pi}\int_{-\infty}^\infty \frac{d\theta_1 d\theta_2 d\theta_3 d\theta_4}{(2\pi)^4} K(\xi)\, e^{2M\max(x_1,x_2)\cosh\xi} e^{-M|r|\sum_i \cosh\theta_i} \\ \cdot\, e^{\mathrm{i}\,\mathrm{sgn}(r)\,M\tau\sum_i \sinh\theta_i} f(\theta_1,\theta_2,\theta_3,\theta_4)\,\langle\theta_4,\theta_3,\theta_2,\theta_1|\,\sigma^z\,|{-\xi},\xi\rangle\,. \tag{C1}$$

As we restrict ourselves to $\tau > 0$ and $r > 0$, we can apply the regularisation scheme (40) to evaluate the second matrix element

$$\begin{aligned}\langle\theta_4,\theta_3,\theta_2,\theta_1|\,\sigma^z\,|{-\xi},\xi\rangle = {}&\langle\theta_4,\theta_3|-\xi,\xi\rangle\, f(\theta_2+\mathrm{i}\,\pi+\mathrm{i}\,\eta_2,\theta_1+\mathrm{i}\,\pi+\mathrm{i}\,\eta_1)\\ &-\langle\theta_4,\theta_2|-\xi,\xi\rangle\, f(\theta_3+\mathrm{i}\,\pi+\mathrm{i}\,\eta_3,\theta_1+\mathrm{i}\,\pi+\mathrm{i}\,\eta_1)\\ &+\langle\theta_4,\theta_1|-\xi,\xi\rangle\, f(\theta_3+\mathrm{i}\,\pi+\mathrm{i}\,\eta_3,\theta_2+\mathrm{i}\,\pi+\mathrm{i}\,\eta_2)\\ &+\langle\theta_3,\theta_2|-\xi,\xi\rangle\, f(\theta_4+\mathrm{i}\,\pi+\mathrm{i}\,\eta_4,\theta_1+\mathrm{i}\,\pi+\mathrm{i}\,\eta_1)\\ &-\langle\theta_3,\theta_1|-\xi,\xi\rangle\, f(\theta_4+\mathrm{i}\,\pi+\mathrm{i}\,\eta_4,\theta_2+\mathrm{i}\,\pi+\mathrm{i}\,\eta_2)\\ &+\langle\theta_2,\theta_1|-\xi,\xi\rangle\, f(\theta_4+\mathrm{i}\,\pi+\mathrm{i}\,\eta_4,\theta_3+\mathrm{i}\,\pi+\mathrm{i}\,\eta_3)\\ &-\langle\theta_4|\xi\rangle\, f(\theta_3+\mathrm{i}\,\pi+\mathrm{i}\,\eta_3,\theta_2+\mathrm{i}\,\pi+\mathrm{i}\,\eta_2,\theta_1+\mathrm{i}\,\pi+\mathrm{i}\,\eta_1,-\xi)\\ &+\langle\theta_3|\xi\rangle\, f(\theta_4+\mathrm{i}\,\pi+\mathrm{i}\,\eta_4,\theta_2+\mathrm{i}\,\pi+\mathrm{i}\,\eta_2,\theta_1+\mathrm{i}\,\pi+\mathrm{i}\,\eta_1,-\xi)\\ &-\langle\theta_2|\xi\rangle\, f(\theta_4+\mathrm{i}\,\pi+\mathrm{i}\,\eta_4,\theta_3+\mathrm{i}\,\pi+\mathrm{i}\,\eta_3,\theta_1+\mathrm{i}\,\pi+\mathrm{i}\,\eta_1,-\xi)\\ &+\langle\theta_1|\xi\rangle\, f(\theta_4+\mathrm{i}\,\pi+\mathrm{i}\,\eta_4,\theta_3+\mathrm{i}\,\pi+\mathrm{i}\,\eta_3,\theta_2+\mathrm{i}\,\pi+\mathrm{i}\,\eta_2,-\xi)\\ &+\langle\theta_4|-\xi\rangle\, f(\theta_3+\mathrm{i}\,\pi+\mathrm{i}\,\eta_3,\theta_2+\mathrm{i}\,\pi+\mathrm{i}\,\eta_2,\theta_1+\mathrm{i}\,\pi+\mathrm{i}\,\eta_1,\xi)\\ &-\langle\theta_3|-\xi\rangle\, f(\theta_4+\mathrm{i}\,\pi+\mathrm{i}\,\eta_4,\theta_2+\mathrm{i}\,\pi+\mathrm{i}\,\eta_2,\theta_1+\mathrm{i}\,\pi+\mathrm{i}\,\eta_1,\xi)\\ &+\langle\theta_2|-\xi\rangle\, f(\theta_4+\mathrm{i}\,\pi+\mathrm{i}\,\eta_4,\theta_3+\mathrm{i}\,\pi+\mathrm{i}\,\eta_3,\theta_1+\mathrm{i}\,\pi+\mathrm{i}\,\eta_1,\xi)\\ &-\langle\theta_1|-\xi\rangle\, f(\theta_4+\mathrm{i}\,\pi+\mathrm{i}\,\eta_4,\theta_3+\mathrm{i}\,\pi+\mathrm{i}\,\eta_3,\theta_2+\mathrm{i}\,\pi+\mathrm{i}\,\eta_2,\xi)\\ &+f(\theta_4+\mathrm{i}\,\pi+\mathrm{i}\,\eta_4,\theta_3+\mathrm{i}\,\pi+\mathrm{i}\,\eta_3,\theta_2+\mathrm{i}\,\pi+\mathrm{i}\,\eta_2,\theta_1+\mathrm{i}\,\pi+\mathrm{i}\,\eta_1,-\xi,\xi),\end{aligned} \tag{C2}$$

This yields three terms, which are graphically represented in Fig. 16. The first one is found to be $C^2_{42}(\tau, x_1, x_2) = C^2_{22}(\tau, x_2, x_1)$ for general x_1 and x_2. Furthermore, the contributions due to lines seven through 14 give

$$C^3_{42}(\tau, x_1, x_2) \;=\; -\frac{\sigma_0^2}{6}\int_{-\infty}^\infty \frac{d\xi}{2\pi}\frac{d\theta_1 d\theta_2 d\theta_3}{(2\pi)^3}\, K(\xi)\prod_{i=1}^3 \tanh\frac{\xi-\theta_i}{2}\,\coth\frac{\xi+\theta_i+\mathrm{i}\,\eta_i}{2} \\ \cdot\prod_{\substack{i,j=1\\ i<j}}^3 \tanh^2\frac{\theta_i-\theta_j}{2}\, e^{2MR\cosh\xi}\, e^{-Mr\sum_i\cosh\theta_i}\, e^{\mathrm{i}\,M\tau(\sinh\xi+\sum_i\sinh\theta_i)} \tag{C3}$$

$$
\begin{aligned}
= \; -\frac{\sigma_0^2}{6}\int_{-\infty}^{\infty}\frac{d\xi}{2\pi}\frac{d\theta_1 d\theta_2 d\theta_3}{(2\pi)^3}\,K\big(\xi+\mathrm{i}\,\tfrac{\pi}{2}\big)\,\prod_{i=1}^{3}\frac{\cosh\xi-\cosh\theta_i}{\cosh\xi+\cosh\theta_i}\\
\cdot\prod_{\substack{i,j=1\\ i<j}}^{3}\tanh^2\frac{\theta_i-\theta_j}{2}\;e^{\mathrm{i}\,2MR\sinh\xi}\;e^{-\mathrm{i}\,Mr\sum_i\sinh\theta_i}\;e^{-M\tau(\cosh\xi+\sum_i\cosh\theta_i)}
\end{aligned}
\tag{C4}
$$

$$
\begin{aligned}
+\Theta(h_{\mathrm{c}}-h)\;\frac{\sigma_0^2}{3}\;\cot v\,\tan\frac{v}{2}\,e^{2MR\cos v}\;e^{-M\tau\sin v}\int_{-\infty}^{\infty}\frac{d\theta_1 d\theta_2 d\theta_3}{(2\pi)^3}\,\prod_{i=1}^{3}\frac{\cosh\theta_i-\sin v}{\cosh\theta_i+\sin v}\\
\cdot\prod_{\substack{i,j=1\\ i<j}}^{3}\tanh^2\frac{\theta_i-\theta_j}{2}\;e^{-\mathrm{i}\,Mr\sum_i\sinh\theta_i}\;e^{-M\tau\sum_i\cosh\theta_i}.
\end{aligned}
\tag{C5}
$$

Here (C4) shows the typical oscillating behaviour and (C5) is present if $h<h_{\mathrm{c}}$.

Finally, the last term of (C2) yields

$$
\begin{aligned}
C_{42}^{4}(\tau,x_1,x_2)\;=-\mathrm{i}\,\frac{\sigma_0^2}{24}\int_0^{\infty}\frac{d\xi}{2\pi}\int_{-\infty}^{\infty}\frac{d\theta_1 d\theta_2 d\theta_3 d\theta_4}{(2\pi)^4}\,K(\xi)\,\tanh\xi\,\prod_{\substack{i,j=1\\ i<j}}^{4}\tanh^2\frac{\theta_i-\theta_j}{2}\\
\cdot\prod_{i=1}^{4}\frac{\cosh\xi+\mathrm{i}\,\sinh\theta_i}{\cosh\xi-\mathrm{i}\,\sinh\theta_i}\,e^{2Mx_2\cosh\xi}\;e^{-\mathrm{i}\,Mr\sum_i\sinh\theta_i}\;e^{-M\tau\sum_i\cosh\theta_i}.
\end{aligned}
\tag{C6}
$$

The next term in the expansion (35) possesses a similar leading term, formally we find $C_{62}^{4}(\tau,x_1,x_2)=C_{42}^{4}(\tau,x_2,x_1)$.

2. Calculation of C_{24}

The fourth to seventh line of (64) yield

$$
\begin{aligned}
C_{24}^{1}(\tau,x_1,x_2)\;=\;\mathrm{i}\,\sigma_0^2\int_0^{\infty}\frac{d\xi_1}{2\pi}\int_{-\infty}^{\infty}\frac{d\xi_2}{2\pi}\frac{d\theta}{2\pi}\,K(\xi_1)\,K\big(\xi_2+\mathrm{i}\,\tfrac{\pi}{2}\big)\,\tanh\xi_1\\
\cdot\frac{\cosh\xi_1-\mathrm{i}\,\sinh\xi_2}{\cosh\xi_1+\mathrm{i}\,\sinh\xi_2}\,\frac{\cosh\xi_1+\mathrm{i}\,\sinh\theta}{\cosh\xi_1-\mathrm{i}\,\sinh\theta}\,\frac{\cosh\xi_2-\cosh\theta}{\cosh\xi_2+\cosh\theta}\\
\cdot e^{2Mx_2\cosh\xi_1}\;e^{\mathrm{i}\,2MR\sinh\xi_2}\;e^{-\mathrm{i}\,Mr\sinh\theta}\;e^{-M\tau(\cosh\theta+\cosh\xi_2)}
\end{aligned}
\tag{C7}
$$

$$
\begin{aligned}
-\Theta(h_{\mathrm{c}}-h)\;2\mathrm{i}\,\sigma_0^2\;\cot v\,\tan\frac{v}{2}\,e^{2MR\cos v}\;e^{-M\tau\sin v}\\
\cdot\int_0^{\infty}\frac{d\xi}{2\pi}\int_{-\infty}^{\infty}\frac{d\theta}{2\pi}\,K(\xi)\,\tanh\xi\,\frac{\cosh\xi-\cos v}{\cosh\xi+\cos v}\frac{\cosh\theta-\sin v}{\cosh\theta+\sin v}\\
\cdot\frac{\cosh\xi+\mathrm{i}\,\sinh\theta}{\cosh\xi-\mathrm{i}\,\sinh\theta}\,e^{2Mx_2\sinh\xi}\;e^{-\mathrm{i}\,Mr\sinh\theta}\;e^{-M\tau\cosh\theta}.
\end{aligned}
\tag{C8}
$$

We stress that the integrand in (C7) is exponentially suppressed for $x_2\to-\infty$. Furthermore, the last term in (64) yields

$$
\begin{aligned}
C_{24}^{2}(\tau,x_1,x_2)=-\,\frac{\sigma_0^2}{4}\int_0^{\infty}\frac{d\xi_1 d\xi_2}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{d\theta_1 d\theta_2}{(2\pi)^2}\prod_{i=1}^{2}K(\xi_i)\,\tanh\xi_i\,\tanh^2\frac{\theta_1-\theta_2}{2}\,\left(\frac{\cosh\xi_1-\cosh\xi_2}{\cosh\xi_1+\cosh\xi_2}\right)^2\\
\cdot\prod_{i,j=1}^{2}\frac{\cosh\xi_i+\mathrm{i}\,\sinh\theta_j}{\cosh\xi_i-\mathrm{i}\,\sinh\theta_j}e^{2Mx_2\sum_i\cosh\xi_i}\;e^{-\mathrm{i}\,Mr\sum_i\sinh\theta_i}\;e^{-M\tau\sum_i\cosh\theta_i}.
\end{aligned}
\tag{C9}
$$

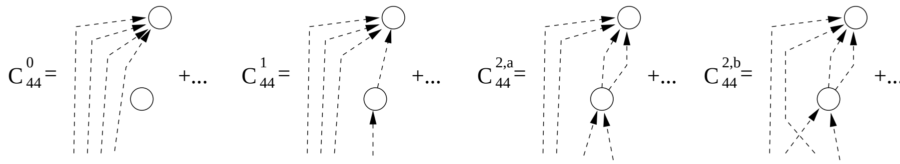

FIG. 17: Graphical representation of C_{44}.

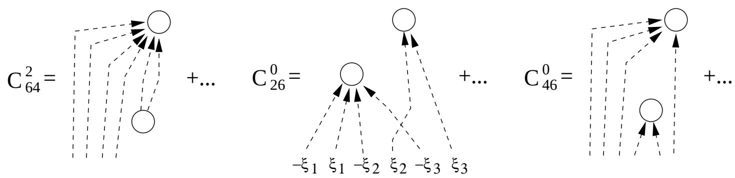

FIG. 18: Graphical representation of C^2_{64}, C^0_{26} and C^0_{46}.

3. Calculation of C_{44}

For the evaluation of C_{44} we have

$$C_{44}(\tau,x_1,x_2)=\frac{1}{48}\int_0^{\infty}\frac{d\xi_1 d\xi_2}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{d\theta_1 d\theta_2 d\theta_3 d\theta_4}{(2\pi)^4}\,K(\xi_1)\,K(\xi_2)\,e^{2Mx_2\sum_i\cosh\xi_i}$$
$$\cdot\, e^{-Mr\sum_i\cosh\theta_i}\,e^{\mathrm{i}\,M\tau\sum_i\sinh\theta_i}\,f(\theta_1,\theta_2,\theta_3,\theta_4)\,\langle\theta_4,\theta_3,\theta_2,\theta_1|\,\sigma^{\mathrm{z}}\,|-\xi_1,\xi_1,-\xi_2,\xi_2\rangle\,. \tag{C10}$$

If we regularise the second matrix element according to (40) we first obtain (see Fig. 17) the completely disconnected term $C^0_{44}(\tau,x_1,x_2)=C_{04}(\tau,x_2,x_1)$. Along the same lines as in the calculation of C^1_{24} we find after some algebra that $C^1_{44}(\tau,x_1,x_2)=C^1_{24}(\tau,x_2,x_1)$. Furthermore, there are two different terms with two lines connecting the two operators, first

$$C^{2,\mathrm{a}}_{44}(\tau,x_1,x_2)=-\frac{\sigma_0^2}{2}\int_0^{\infty}\frac{d\xi_1 d\xi_2}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{d\theta_1 d\theta_2}{(2\pi)^2}\prod_{i=1}^{2}K(\xi_i)\,\tanh\xi_i\prod_{i=1}^{2}\frac{\cosh\xi_1-\mathrm{i}\,\sinh\theta_i}{\cosh\xi_1+\mathrm{i}\,\sinh\theta_i}\,\frac{\cosh\xi_2+\mathrm{i}\,\sinh\theta_i}{\cosh\xi_2-\mathrm{i}\,\sinh\theta_i}$$
$$\cdot\,\tanh^2\frac{\theta_1-\theta_2}{2}\,e^{2Mx_1\cosh\xi_1}\,e^{2Mx_2\cosh\xi_2}\,e^{-\mathrm{i}\,Mr\sum_i\sinh\theta_i}\,e^{-M\tau\sum_i\cosh\theta_i}, \tag{C11}$$

which is not symmetric under $\xi_1\leftrightarrow\xi_2$, and second,

$$C^{2,\mathrm{b}}_{44}(\tau,x_1,x_2)=-\frac{\sigma_0^2}{4}\int_{-\infty}^{\infty}\frac{d\xi_1 d\xi_2}{(2\pi)^2}\frac{d\theta_1 d\theta_2}{(2\pi)^2}\,K(\xi_1)\,K(\xi_2)\,\tanh^2\frac{\xi_1-\xi_2}{2}\,\tanh^2\frac{\theta_1-\theta_2}{2}$$
$$\cdot\prod_{i,j=1}^{2}\frac{\cosh\theta_j+\mathrm{i}\,\sinh\xi_i}{\cosh\theta_j-\mathrm{i}\,\sinh\xi_i}\,e^{2MR\sum_i\cosh\xi_i}\,e^{-\mathrm{i}\,Mr\sum_i\sinh\theta_i}\,e^{\mathrm{i}\,M\tau\sum_i\sinh\xi_i}\,e^{-M\tau\sum_i\cosh\theta_i}. \tag{C12}$$

Here we can again shift $\xi_{1,2}\to\xi_{1,2}+\mathrm{i}\,\pi/2$. The result will, however, only yield a negligible correction to the spectral function.

4. Calculation of C_{64}, C_{26} and C_{46}

The last terms of the series (35) we wish to calculate are the leading contributions of C_{64}, C_{26} and C_{46} sketched in Fig. 18. The first one is found to be $C^2_{64}(\tau,x_1,x_2)=C^2_{24}(\tau,x_2,x_1)$.

For the evaluation of the second one we start with

$$C_{26}(\tau,x_1,x_2)=\frac{1}{12}\int_0^{\infty}\frac{d\xi_1 d\xi_2 d\xi_3}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{d\theta_1 d\theta_2}{(2\pi)^2}\prod_{i=1}^{3}K(\xi_i)\,e^{2Mx_2\sum_i\cosh\xi_i}$$
$$\cdot\, e^{-Mr\sum_i\cosh\theta_i}\,e^{\mathrm{i}\,M\tau\sum_i\sinh\theta_i}\,f(\theta_1,\theta_2)\,\langle\theta_2,\theta_1|\,\sigma^{\mathrm{z}}\,|-\xi_1,\xi_1,-\xi_2,\xi_2,-\xi_3,\xi_3\rangle\,. \tag{C13}$$

If we regularise the second matrix element according to (40) and keep only those terms in which the two intermediate particles are contracted with two particles possessing different rapidities from the boundary state, *e.g.* the term proportional to $\langle\theta_2,\theta_1|\,\xi_2,\xi_3\rangle$, we find

$$
\begin{aligned}
C^0_{26}(\tau,x_1,x_2) \;=\;& \mathrm{i}\,\frac{\sigma_0^2}{2}\int_0^\infty\frac{d\xi_1}{2\pi}\,K(\xi_1)\,\tanh\xi_1\,e^{2Mx_2\cosh\xi_1}\\
&\cdot\int_{-\infty}^{\infty}\frac{d\xi_2 d\xi_3}{(2\pi)^2}\,K\big(\xi_2+\mathrm{i}\,\tfrac{\pi}{2}\big)\,K\big(\xi_3+\mathrm{i}\,\tfrac{\pi}{2}\big)\,\tanh^2\frac{\xi_2-\xi_3}{2}\\
&\cdot\prod_{i=2}^{3}\frac{\cosh\xi_1-\mathrm{i}\,\sinh\xi_i}{\cosh\xi_1+\mathrm{i}\,\sinh\xi_i}\,e^{\mathrm{i}\,2MR\sum_{i=2}^3\sinh\xi_i}\,e^{-M\tau\sum_{i=2}^3\cosh\xi_i}
\end{aligned}
\tag{C14}
$$

$$
\begin{aligned}
&+\Theta(h_\mathrm{c}-h)\;2\mathrm{i}\,\sigma_0^2\,\cot v\,\tan\frac{v}{2}\,e^{2MR\cos v}\,e^{-M\tau\sin v}\\
&\cdot\int_0^\infty\frac{d\xi_1}{2\pi}\int_{-\infty}^{\infty}\frac{d\xi_2}{2\pi}\,K(\xi_1)\,K\big(\xi_2+\mathrm{i}\,\tfrac{\pi}{2}\big)\,\tanh\xi_1\,e^{2Mx_2\cosh\xi_1}\,\frac{\cosh\xi_1-\cos v}{\cosh\xi_1+\cos v}\\
&\cdot\tanh^2\frac{\xi_2+\mathrm{i}\,(\pi/2-v)}{2}\,\frac{\cosh\xi_1-\mathrm{i}\,\sinh\xi_2}{\cosh\xi_1+\mathrm{i}\,\sinh\xi_2}\,e^{\mathrm{i}\,2MR\sinh\xi_2}\,e^{-M\tau\cosh\xi_2}.
\end{aligned}
\tag{C15}
$$

We label (C14) by $C_{26}^{0,\mathrm{osc}}$ and (C15) by $C_{26}^{0,\mathrm{bbs}}$. In the same way one obtains $C^0_{46}(\tau,x_1,x_2)=C^0_{26}(\tau,x_2,x_1)$.

APPENDIX D: HIGHER-ORDER CORRECTIONS TO THE SPECTRAL FUNCTION

1. Two- and four-particle continuum

The second-order contribution to the two-particle continuum is given by $C^2_{24}+C^{2,\mathrm{a}}_{44}+C^2_{64}$, which after Fourier transformation (73) and analytic continuation reads

$$
\begin{aligned}
C^2_{2K}(\omega,R,r=0)&=\big(C^2_{24}+C^{2,\mathrm{a}}_{44}+C^2_{64}\big)(\omega,R,r=0)\\
&=\frac{\sigma_0^2}{2}\int_0^\infty\frac{d\xi_1d\xi_2}{(2\pi)^2}\int_{-\infty}^{\infty}\frac{d\theta_1d\theta_2}{(2\pi)^2}\prod_{i=1}^{2}K(\xi_i)\,\tanh\xi_i\,\frac{e^{2MR\sum_i\cosh\xi_i}}{\omega-M\sum_i\cosh\theta_i+\mathrm{i}\,\delta}\\
&\quad\cdot\tanh^2\frac{\theta_1-\theta_2}{2}\left[\left(\frac{\cosh\xi_1-\cosh\xi_2}{\cosh\xi_1+\cosh\xi_2}\right)^2\prod_{i,j=1}^{2}\frac{\cosh\xi_i+\mathrm{i}\,\sinh\theta_j}{\cosh\xi_i-\mathrm{i}\,\sinh\theta_j}\right.\\
&\quad\left.+\prod_{i=1}^{2}\frac{\cosh\xi_1-\mathrm{i}\,\sinh\theta_i}{\cosh\xi_1+\mathrm{i}\,\sinh\theta_i}\,\frac{\cosh\xi_2+\mathrm{i}\,\sinh\theta_i}{\cosh\xi_2-\mathrm{i}\,\sinh\theta_i}\right].
\end{aligned}
\tag{D1}
$$

When calculating the spectral function, the contributions proportional to the principal value vanish as the imaginary part of the terms in the squared brackets is antisymmetric under $\theta_i\to-\theta_i$. After some straightforward algebra we obtain

$$
\begin{aligned}
S^2_{2K}(\omega,R,r=0)=\sigma_0^2&\int_0^\infty\frac{d\xi_1d\xi_2}{(2\pi)^2}\int_{-\theta'}^{\theta'}\frac{d\theta}{2\pi}\prod_{i=1}^{2}\frac{\hat{K}(\xi_i)\,\tanh\xi_i}{(\cosh^2\xi_i+\sinh^2\theta)(\cosh^2\xi_i+\sinh^2\tilde{\theta})}\,\frac{\tanh^2\frac{\theta-\tilde{\theta}}{2}}{\sqrt{(\omega-M\cosh\theta)^2-M^2}}\\
\cdot\frac{e^{2MR\sum_i\cosh\xi_i}}{(\cosh\xi_1+\cosh\xi_2)^2}&\left\{\big(\cosh^2\xi_1+\cosh^2\xi_2\big)\prod_{i=1}^{2}\Big[\big(\cosh^2\xi_i-\sinh^2\theta\big)\big(\cosh^2\xi_i-\sinh^2\tilde{\theta}\big)-4\cosh^2\xi_i\,\sinh\theta\,\sinh\tilde{\theta}\Big]\right.\\
&\left.+\,8\prod_{i=1}^{2}\cosh\xi_i\,\Big[\big(\cosh^2\xi_2-\sinh^2\theta\big)\sinh\tilde{\theta}+\big(\cosh^2\xi_i-\sinh^2\tilde{\theta}\big)\sinh\theta\Big]\right\},
\end{aligned}
\tag{D2}
$$

where $\tilde{\theta}$ and θ' are defined in (78). We stress that the integrand is exponentially suppressed for large $M|R|$ by the factor $e^{2MR\sum_{i=1}^{2}\cosh\xi_i}$.

The first order correction in the boundary reflection matrix to the four-particle continuum is given by $C_{42}^4 + C_{62}^4$. The corresponding spectral function is

$$
\begin{aligned}
S_{1K}^4(\omega, R, r=0) &= \big(S_{42}^4 + S_{62}^4\big)(\omega, R, r=0) \\
&= \frac{\sigma_0^2}{12M}\int_0^\infty \frac{d\xi}{2\pi}\int_{\mathcal{A}(\omega)} \frac{d\theta_1 d\theta_2 d\theta_3}{(2\pi)^3}\, \frac{\hat{K}(\xi)\,\tanh\xi}{\cosh^2\xi + \sinh^2\tilde{\theta}_4}\, \frac{e^{2MR\cosh\xi}}{\sinh\tilde{\theta}_4} \\
&\quad \cdot \prod_{i<j}^{3}\tanh^2\frac{\theta_i - \theta_j}{2}\,\prod_{i=1}^{3}\tanh^2\frac{\theta_i - \tilde{\theta}_4}{2}\,\prod_{i=1}^{3}\frac{1}{\cosh^2\xi + \sinh^2\theta_i} \\
&\quad \cdot \Bigg\{\big(\cosh^2\xi - \sinh^2\tilde{\theta}_4\big)\Bigg[\prod_{i=1}^{3}\big(\cosh^2\xi - \sinh^2\theta_i\big) - 4\cosh^2\xi\,\sum_{i=1}^{3}\big(\cosh^2\xi - \sinh^2\theta_i\big)\prod_{j\neq i}^{3}\sinh\theta_j\Bigg] \\
&\qquad + 4\cosh^2\xi\,\sinh\tilde{\theta}_4\Bigg[4\cosh^2\xi\,\prod_{i=1}^{3}\sinh\theta_i - \sum_{i=1}^{3}\sinh\theta_i\prod_{j\neq i}^{3}\big(\cosh^2\xi - \sinh^2\theta_j\big)\Bigg]\Bigg\},
\end{aligned}
\tag{D3}
$$

where $\tilde{\theta}_4$ and $\mathcal{A}(\omega)$ are defined by

$$
\tilde{\theta}_4 = \mathrm{Arcosh}\Big(\frac{\omega}{M} - \sum\nolimits_i^3 \cosh\theta_i\Big),\; \mathcal{A}(\omega) = \Big\{(\theta_1,\theta_2,\theta_3)\in\mathbb{R}^3 \Big| \sum\nolimits_i^3 \cosh\theta_i \le \frac{\omega}{M} - 1\Big\}. \tag{D4}
$$

2. Oscillating terms

The first correction to (80) is given by $C_{24}^{1,\mathrm{osc}} + C_{44}^{1,\mathrm{osc}}$. The usual steps yield for the corresponding spectral function

$$
\begin{aligned}
&\big(S_{24}^{1,\mathrm{osc}} + S_{44}^{1,\mathrm{osc}}\big)(\omega, R, r=0) \\
&= 2\sigma_0^2\int_0^\infty \frac{d\xi}{2\pi}\int_{-\theta'}^{\theta'}\frac{d\theta}{2\pi}\,\frac{\hat{K}(\xi)\,\tanh\xi}{\cosh^2\xi + \sinh^2\tilde{\theta}}\,\frac{\cosh\theta - \cosh\tilde{\theta}}{\cosh\theta + \cosh\tilde{\theta}}\,\frac{\cosh^2\xi - \sinh^2\theta}{\cosh^2\xi + \sinh^2\theta}\,\frac{e^{2MR\cosh\xi}}{\sqrt{(\omega - M\cosh\theta)^2 - M^2}} \\
&\quad \cdot \Big\{\Big[\mathfrak{Re}K\big(\tilde{\theta} + \mathrm{i}\tfrac{\pi}{2}\big)\,\big(\cosh^2\xi - \sinh^2\tilde{\theta}\big) + 2\,\mathfrak{Im}K\big(\tilde{\theta} + \mathrm{i}\tfrac{\pi}{2}\big)\,\cosh\xi\,\sinh\tilde{\theta}\Big]\,\cos\big(2MR\sinh\tilde{\theta}\big) \\
&\qquad - \Big[\mathfrak{Im}K\big(\tilde{\theta} + \mathrm{i}\tfrac{\pi}{2}\big)\,\big(\cosh^2\xi - \sinh^2\tilde{\theta}\big) - 2\,\mathfrak{Re}K\big(\tilde{\theta} + \mathrm{i}\tfrac{\pi}{2}\big)\,\cosh\xi\,\sinh\tilde{\theta}\Big]\,\sin\big(2MR\sinh\tilde{\theta}\big)\Big\},
\end{aligned}
\tag{D5}
$$

where $\tilde{\theta}$ and θ' are defined in (78). We stress that in comparison to (80) there appears an extra factor $\hat{K}(\xi)\, e^{2MR\cosh\xi}$, which shows that (D5) is indeed the first correction in K to (80). The first correction in K to (82) is given by

$$
\begin{aligned}
&\big(S_{26}^{0,\mathrm{osc}} + S_{46}^{0,\mathrm{osc}}\big)(\omega, R, r=0) \\
&\quad = -\sigma_0^2 \int_0^\infty \frac{d\xi}{2\pi} \int_{-\theta'}^{\theta'} \frac{d\theta}{2\pi}\, \frac{\hat{K}(\xi)\, \tanh\xi}{\cosh^2\xi + \sinh^2\theta}\, \frac{e^{2MR\cosh\xi}}{\cosh^2\xi + \sinh^2\tilde{\theta}}\, \frac{\tanh^2\frac{\theta-\tilde{\theta}}{2}}{\sqrt{(\omega - M\cosh\theta)^2 - M^2}} \\
&\qquad \cdot \Bigg\{ \bigg[\Big(\mathfrak{Re}K\big(\theta + \mathrm{i}\,\tfrac{\pi}{2}\big)\, \mathfrak{Re}K\big(\tilde{\theta} + \mathrm{i}\,\tfrac{\pi}{2}\big) - \mathfrak{Im}K\big(\theta + \mathrm{i}\,\tfrac{\pi}{2}\big)\, \mathfrak{Im}K\big(\tilde{\theta} + \mathrm{i}\,\tfrac{\pi}{2}\big) \Big) \\
&\qquad\qquad \cdot \Big(\big(\cosh^2\xi - \sinh^2\theta\big)\big(\cosh^2\xi - \sinh^2\tilde{\theta}\big) - 4\cosh^2\xi\, \sinh\theta\, \sinh\tilde{\theta} \Big) \\
&\qquad\quad - 4\,\mathfrak{Re}K\big(\theta + \mathrm{i}\,\tfrac{\pi}{2}\big)\, \mathfrak{Im}K\big(\tilde{\theta} + \mathrm{i}\,\tfrac{\pi}{2}\big)\, \cosh\xi \Big(\big(\cosh^2\xi - \sinh^2\theta\big)\sinh\tilde{\theta} + \big(\cosh^2\xi - \sinh^2\tilde{\theta}\big)\sinh\theta \Big) \bigg] \\
&\qquad\qquad\qquad \cdot \cos\big(2MR(\sinh\theta + \sinh\tilde{\theta})\big) \\
&\qquad + 2\bigg[\Big(\mathfrak{Im}K\big(\theta + \mathrm{i}\,\tfrac{\pi}{2}\big)\, \mathfrak{Im}K\big(\tilde{\theta} + \mathrm{i}\,\tfrac{\pi}{2}\big) - \mathfrak{Re}K\big(\theta + \mathrm{i}\,\tfrac{\pi}{2}\big)\, \mathfrak{Re}K\big(\tilde{\theta} + \mathrm{i}\,\tfrac{\pi}{2}\big) \Big) \cosh\xi \\
&\qquad\qquad \cdot \Big(\big(\cosh^2\xi - \sinh^2\theta\big)\sinh\tilde{\theta} + \big(\cosh^2\xi - \sinh^2\tilde{\theta}\big)\sinh\theta \Big) \\
&\qquad\qquad - \mathfrak{Re}K\big(\theta + \mathrm{i}\,\tfrac{\pi}{2}\big)\, \mathfrak{Im}K\big(\tilde{\theta} + \mathrm{i}\,\tfrac{\pi}{2}\big) \Big(\big(\cosh^2\xi - \sinh^2\theta\big)\big(\cosh^2\xi - \sinh^2\tilde{\theta}\big) - 4\cosh^2\xi\, \sinh\theta\, \sinh\tilde{\theta} \Big) \bigg] \\
&\qquad\qquad\qquad \cdot \sin\big(2MR(\sinh\theta + \sinh\tilde{\theta})\big) \Bigg\}.
\end{aligned}
\tag{D6}
$$

The oscillating contribution to local spectral function to second order in K is given by

$$
S_{2K}^{\mathrm{osc}}(\omega, R, r) = S_{24}^{1,\mathrm{osc}}(\omega, R, r) + S_{44}^{1,\mathrm{osc}}(\omega, R, r) + S_{26}^{0,\mathrm{osc}}(\omega, R, r) + S_{46}^{0,\mathrm{osc}}(\omega, R, r). \tag{D7}
$$

We observe that the integrands are exponentially suppressed for large distances from the boundary.

3. Contributions from the boundary bound state

The second-order contribution of the boundary bound state to the spectral function is given by

$$
S_{2K}^{\mathrm{bbs}}(\omega, R, r) = S_{24}^{1,\mathrm{bbs}}(\omega, R, r) + S_{44}^{1,\mathrm{bbs}}(\omega, R, r) + S_{26}^{0,\mathrm{bbs}}(\omega, R, r) + S_{46}^{0,\mathrm{bbs}}(\omega, R, r), \tag{D8}
$$

where

$$
\begin{aligned}
\big(S_{24}^{1,\mathrm{bbs}} + S_{44}^{1,\mathrm{bbs}}\big)(\omega, R, r=0) &= 4\sigma_0^2\, \cot v\, \tan\frac{v}{2}\, e^{2MR\cos v} \\
&\quad \cdot \int_0^\infty \frac{d\xi}{2\pi}\, \frac{\hat{K}(\xi)\, \tanh\xi\, e^{2MR\cosh\xi}}{\sqrt{(\omega - M\sin v)^2 - M^2}}\, \frac{\cosh\xi - \cos v}{\cosh\xi + \cos v}\, \frac{\cosh\hat{\xi} - \sin v}{\cosh\hat{\xi} + \sin v}\, \frac{\cosh^2\xi - \sinh^2\hat{\xi}}{\cosh^2\xi + \sinh^2\hat{\xi}}
\end{aligned}
\tag{D9}
$$

and

$$
\begin{aligned}
\left(S_{26}^{0,\mathrm{bbs}}+S_{46}^{0,\mathrm{bbs}}\right)(\omega,R,r=0) &= -\frac{4M^2\sigma_0^2}{\omega^2}\,\cot v\,\tan\frac{v}{2}\,e^{2MR\cos v}\\
&\cdot\int_0^\infty\frac{d\xi}{2\pi}\,\frac{\hat{K}(\xi)\,\tanh\xi\,e^{2MR\cosh\xi}}{\sqrt{(\omega-M\sin v)^2-M^2}}\,\frac{\cosh\xi-\cos v}{\cosh\xi+\cos v}\,\frac{1}{\cosh^2\xi+\sinh^2\hat{\xi}}\\
&\quad\cdot\Bigg\{\bigg[\mathfrak{Re}K\big(\hat{\xi}+\mathrm{i}\,\tfrac{\pi}{2}\big)\Big[\big(\cosh 2\hat{\xi}-\cos 2v-2\big)\,\big(\cosh^2\xi-\sinh^2\hat{\xi}\big)/2+4\,\cosh\xi\,\sinh^2\hat{\xi}\,\cos v\Big]\\
&\qquad+\mathfrak{Im}K\big(\hat{\xi}+\mathrm{i}\,\tfrac{\pi}{2}\big)\,\sinh\hat{\xi}\Big[\big(\cosh 2\hat{\xi}-\cos 2v-2\big)\,\cosh\xi-2\,\cos v\,\big(\cosh^2\xi-\sinh^2\hat{\xi}\big)\Big]\bigg]\\
&\qquad\qquad\cdot\cos\big(2MR\sinh\hat{\xi}\big)\\
&\quad-\bigg[\mathfrak{Re}K\big(\hat{\xi}+\mathrm{i}\,\tfrac{\pi}{2}\big)\,\sinh\hat{\xi}\Big[2\,\cos v\,\big(\cosh^2\xi-\sinh^2\hat{\xi}\big)-\big(\cosh 2\hat{\xi}-\cos 2v-2\big)\,\cosh\xi\Big]\\
&\qquad+\mathfrak{Im}K\big(\hat{\xi}+\mathrm{i}\,\tfrac{\pi}{2}\big)\Big[\big(\cosh 2\hat{\xi}-\cos 2v-2\big)\,\big(\cosh^2\xi-\sinh^2\hat{\xi}\big)/2+4\,\cosh\xi\,\sinh^2\hat{\xi}\,\cos v\Big]\bigg]\\
&\qquad\qquad\cdot\sin\big(2MR\sinh\hat{\xi}\big)\Bigg\}.
\end{aligned}
\tag{D10}
$$

Here $\hat{\xi}$ is given in (88).

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